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ABSTRACT

This text presents lessons in specific mathematical concepts. It usually presents concepts in the context of problems typically occurring in health care situations or relating to biomedical concepts. Some lessons relate to concepts being studied in other disciplinary components of the biomedical interdisciplinary curriculum. For example, symbolic logic is presented in the context of computer science presented in the science portion of the curriculum. (RE)

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BIOMEDICAL MATHEMATICS

UNIT IV

SYMBOLIC LOGIC, TRIGONOMETRY AND STATISTICS

STUDENT TEXT

REVISED VERSION, 1976

THE BIOMEDICAL INTERDISCIPLINARY CURRICULUM PROJECT

SUPPORTED BY THE NATIONAL SCIENCE FOUNDATION

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SECTION 1: LOGICAL STATEMENTS

1-1 What is a Logical Statement?

In the next several sections you will be learning about a subject known as symbolic logic. One of our purposes is to give you the mathematical skills you will need to understand the electrical circuits you will soon be working with in the science class. There, you will design and construct circuits that will diagnose diseases when you enter information on a patient's symptoms.

These circuits are, in fact, simple examples of the type of circuits used in computers. The same techniques of symbolic logic which you will use in designing your circuits are used on a larger scale to design many of the circuits used in large high-speed computers.

This first section deals with the logical concept of statement. We will consider a certain class of sentences, called statements, and we will give a definition of the concept statement that will enable us to decide whether or not any given sentence is a statement. We will also explain how to translate a statement into symbolic form.

Consider the following sentences:

1. The patient's temperature is above 38 °C.
2. Yesterday, at the airport weather station, there was measurable precipitation.
3. It generally rains on Monday at Pudworthy High School.
4. Elmo often goes to the hamburger stand after school.

How would we decide whether Sentence 1 is true? Obviously all we have to do is to find a thermometer and take the patient's temperature. How would we decide whether Sentence 2 is true? We could just check with the weather station. Clearly, we can decide whether sentences 1 and 2 are true by using simple, well-defined procedures.

Sentences 3 and 4 are different. Does it generally rain on Monday? That depends on what we mean by generally. Does generally mean more than half the time? Nine times out of ten? Thirty per cent of the time? If we knew exactly what generally meant, we could decide the truth or falsity of Sentence 3 in much the same way as we decide the truth or falsity of Sentence 2. Similarly, we cannot decide whether Sentence 4 is true or false unless we know exactly what is meant by often.

A large part of our study of symbolic logic will be concerned with deciding whether a given sentence is true or false. Obviously, it wouldn't make sense to talk about such matters if we didn't have well-established ways of deciding whether or not sentences are true.

Therefore, in our study of symbolic logic, we will work only with sentences which can be verified (that is, checked for truth or falsity) in a well-defined way, such as Sentences 1 and 2. We will ignore sentences such as 3 and 4.

This property of "being verifiable by well-defined means" is important enough to deserve a definition.

DEFINITION: A <u>logical statement</u> is a proposition which has a precisely defined meaning; that is, it is possible to decide in a well-defined way whether the proposition is true or false. A logical statement uses only terms which have a well-established, agreed-on meaning.

Notice that this definition of logical statement is much more restrictive than ordinary usage. Sentences 1 and 2 are logical statements, while Sentences 3 and 4 are not. But in ordinary usage we would consider all four sentences to be perfectly good statements, even though we would probably admit that the last two are more vague.

1-2 <u>More Examples</u>

Now consider the following propositions.

1. The patient's pulse is less than 60 beats per minute.
2. Several Pudworthy High School students were out sick last Thursday.
3. Either Lois is 21 or Fred is 101 or both.
4. The distance to the nearest hospital is about six miles.
5. At least one of the following holds.
   Hortense has blood type A.
   Hortense has eleven fingers.
   Hortense owns a Volkswagen.
6. The patient has a strep throat.

Let us decide which of these six propositions are logical statements. It is easy to decide the truth or falsity of Proposition 1. We need only take the patient's pulse. Therefore <u>Proposition 1 is a logical statement</u>.

Look at Proposition 2. There is no well-defined way of deciding the truth or falsity of this proposition unless we can determine the exact meaning of <u>several</u>. So <u>Proposition 2 is not a logical statement</u>.

Proposition 3 is a bit more complicated because it depends on two separate assertions. However, Lois and Fred's ages can both be determined by straightforward means. Once this is done we can easily decide on truth or falsity. So <u>Proposition 3 is a statement</u>.

Proposition 4 is another case of vague terminology. How do you decide whether or not the distance to the hospital is <u>about</u> six miles? <u>Proposition 4 is not a statement</u>, according to our definition.

Now consider Proposition 5. Does Hortense have blood type A? We can find out simply by testing her blood. Does she have eleven fingers, and does she own a Volkswagen? We can check these assertions also. Then if one or more of the three assertions is true, Proposition 5 is also true. Otherwise, Proposition 5 is false.

Thus Proposition 5 satisfies our definition of a statement, even though it is not written in the form of a single sentence.

Proposition 6 is the sort of assertion you will see often in future sections. The truth or falsity of this claim must be decided by a diagnosis. In borderline cases there might be confusion in deciding whether a patient has strep throat or some similar infectious disease. To simplify matters, we will assume that the doctor in question has a procedure for making the decision. Therefore we will treat Proposition 6 as a logical statement.

1-3 A Shorthand for Statements

All the statements (and non-statements) we have considered above are propositions written in ordinary language. However, in the study of symbolic logic, we will usually work with statements which have been written in symbolic form. Using special symbols as a kind of shorthand, we can find concise forms of statements which are very complex when written in plain English. Also, we will later develop methods of manipulating statements and performing calculations with them. This will be easier to do if the statements are written in symbols, rather than words. This, incidentally, is the essense of computer programming--converting statements into symbols.

Our system of symbols is very simple. We will substitute letters for basic statements, and special symbols for the words "and," "or" and "not." This is not very different from the system of symbols we use in algebra, where we substitute letters for arbitrary or unknown numbers and use the special symbols +, = and - for the words plus, equals and minus (or negative).

For example, let the letters p, q and r stand for the following basic statements.

p: The patient has chicken pox.
q: The patient has measles.
r: The patient has a cold.

Our special symbols are $\wedge$ for and, $\vee$ for or and $^{-}$ (a raised bar) for not.

Then the statement, "The patient does not have chicken pox," (that is, not p) would be written $\overline{p}$. If we wished to state, "The patient has chicken pox and measles," (that is, p and q) we would write $p \wedge q$. And the statement, "The patient has measles or a cold," (that is, q or r) is written $q \vee r$.

1-4 A Problem With the Word "or"

Before going on to other examples, we must clear up an uncertainty in the meaning of the word or. In English, the word or is used in two different ways. Consider the following two sentences.

1. Elmo might have trouble with problem 1 or problem 2.
2. Elmo got an A or a B on the last exam.

Sentence 1 certainly says that Elmo might have trouble with one of the two problems, but it does not disallow that he might have trouble with both. There is an implied "or both" in this sentence. Sentence 2 is different. Elmo cannot have gotten both an A and a B on the last exam. So this sentence contains an implied "but not both."

Sentence 1 is an example of the inclusive or (so called, because it includes the possibility of both being true). Sentence 2 is an example of the exclusive or (so called, because it excludes the possibility of both being true). In ordinary English, the word or is used in the exclusive sense (as in Sentence 2) at least as often as in the inclusive sense (as in Sentence 1). However, it turns out that the mathematics of our symbolic logic is simpler if we use the inclusive or as our basic translation for the word or.

Thus, when we write $p \vee q$, we mean p (inclusive) or q, that is, p or q (or both). The exclusive case p or q (but not both) is not written $p \vee q$. In fact, it is written $(p \wedge \overline{q}) \vee (\overline{p} \wedge q)$; however we are not yet ready to show why this is so.

Let us look at some more examples, using the same basic statements p, q, r as in the preceding examples. If we wish to say, "The patient has chicken pox and does not have a cold," we would write $p \wedge \overline{r}$. To say, "The patient has measles or has no cold (or both)," we would write $q \vee \overline{r}$. To say, "The patient has chicken pox and measles, and does not have a cold," we would write $p \wedge q \wedge \overline{r}$.

The symbol $\overline{}$ is not limited to basic statements; it can be applied to any statement. For instance, "It is not true that the patient has chicken pox and does not have a cold," is written $\overline{p \wedge \overline{r}}$. And the statement, "it is not true that the patient does not have chicken pox," is written $\overline{\overline{p}}$.

Sometimes we find a sentence in ordinary English which has more than one meaning, depending on how we group the words and and or. Consider the sentence, "The patient has chicken pox or measles and a cold." This sentence could be understood in two different ways.

MEANING 1. One or both of the following is true:

a. The patient has chicken pox.

b. The patient has measles and a cold.

We can use a comma to show this meaning. "The patient has chicken pox, or measles and a cold." Here, the word or is applied to the statement p, and to the entire phrase, "the patient has measles and a cold" (in symbols, $q \wedge r$). The word and is applied only to the statements q and r. In symbols, we write $p \vee (q \wedge r)$.

MEANING 2: Both of the following are true:

a. The patient has chicken pox or measles (or both).

b. The patient has a cold.

We can use a comma to convey this meaning as follows. "The patient has chicken pox or measles, and a cold." Here, the word or applies only to the statements p and q, but the word and applies to the statement r and to the phrase, "The patient has chicken pox or measles (or both)" (in symbols, $p \vee q$). Thus Meaning 2 is written $(p \vee q) \wedge r$.

Our two meanings are written with the same symbols p, q, r, $\vee$ and $\wedge$ in the same order. But we use parentheses to group these symbols in two different ways, just as we use a comma to convey the proper meaning in sentences. If no comma appears in the sentence or no parentheses in the logical statement, then either of the two meanings is possible and there is no way to choose between them.

It is easier to understand this use of parentheses if we look at a parallel example from the number system. For example, the expression

$$3 \times 4 + 5$$

can have two different meanings, depending on how the numbers are grouped.

$$3 \times (4 + 5) = 27$$
$$(3 \times 4) + 5 = 17$$

There is another parallel between numbers and statements. If only <u>one</u> of the operations, + or x appears in an expression, then the answer is the same regardless of how parentheses are introduced. For example, both of the following are equal to 15.

$$(1 + 2 + 3) + (4 + 5)$$
$$(1 + 2) + (3 + 4) + 5$$

Similarly, if only one of the operations $\wedge$, $\vee$ appear in a statement then the placement of parentheses does not change the meaning of the statement. Both of the following represent the same statement.

$$(p \vee q) \vee (r \vee s)$$
$$p \vee (q \vee r) \vee s$$

PROBLEM SET 1:

For Problems 1 through 11, if the proposition is a logical statement write "S." Otherwise write "NS."

1. Horace made a score of 69 on the last test.
2. Strephon almost always scores over 50 on math tests.
3. Elmo usually eats hot dogs at breakfast.
4. Last March 31 the temperature went above 61 °F at the local airport.
5. Mrs. Abernathy has blood type AB.
6. The patient weighs approximately 286 pounds.
7. At least 38 people attended the last birdwatchers' field trip.
8. This ostrich egg weighs under 3 kilograms.
9. Both of the following are true.
   a. Hortense lives with a gorilla.
   b. Hortense is a gorilla.
10. At least one of the following is true.
   a. Gloria's birthday is September 13.
   b. Alice's birthday is December 31.

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11. Both of the following are false.
   a. The patient has normal reflexes.
   b. The patient has a normal electroencephalogram.

Suppose that the three statements below are given.

p: Paul is pale.
q: Quincy is queasy.
r: Robert is robust.

Put each of the following statements in symbolic language.

12. Paul is pale or Robert is robust. (From now on we will omit writing the "or both," but you should always assume that it is meant to be part of any sentence with "or" in it.)

13. Paul is pale and Quincy is queasy.

14. Paul is pale and Paul is not pale.

15. Quincy is queasy or Quicy is not queasy.

16. Paul is not pale or Quincy is not queasy.

17. Paul is pale and Quincy is queasy and Robert is robust.

18. Paul is pale or Quincy is not queasy or Robert is not robust.

19. Paul is pale, and either Quincy is queasy or Robert is robust.

20. The following is not true: Quincy is queasy or Robert is robust.

21. At least one of the following holds.
   a. Paul is not pale.
   b. This is not true: Quincy is queasy and Robert is robust.

Suppose that the three statements below are given.

p = Strephon has strep
q = John has jaundice
r = Rick has rickets

Translate the following symbolic statements into written sentences.

22. $p \vee q$	24. $\overline{q} \wedge r$	26. $(p \wedge \overline{q}) \vee r$
23. $\overline{r}$	25. $\overline{p} \wedge q \wedge \overline{r}$	27. $(p \vee \overline{q}) \wedge (\overline{p} \vee r)$

SECTION 2: TRUTH TABLES

2-1 Introduction

In the last section we defined a logical statement to be a proposition which has a precisely defined meaning. When we say a given proposition is a statement, we mean that there is some way to tell whether it is true or false.

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We also can combine several basic statements to form more complex statements by using the concepts and, or and not (in symbols, $\wedge$, $\vee$, $\bar{\ }$). In general, the truth or falsity of the basic statements will determine the truth or falsity of the complex statement. For example, the statement $p \wedge q$ cannot be true unless both p and q are true; if either p or q is false, then $p \wedge q$ is also false.

In this lesson we will see how to construct a truth table which shows clearly how the truth or falsity of a complex statement depends on the truth or falsity of the basic statements from which it was constructed. Such tables are necessary for constructing the circuits you will use to diagnose diseases.

2-2 A Truth Table for $\bar{p}$

Consider the following basic statement:

p: The patient has hepatitis.

Then the statement $\bar{p}$ is, "The patient does not have hepatitis." Statement $\bar{p}$ will be true when statement p is false, and false when statement p is true. So we can construct the following truth table for $\bar{p}$.

Truth Table for $\bar{p}$

p	$\bar{p}$
true	false
false	true

This table is easy to interpret. On the left side, under the heading p, are listed the two possible cases for the truth or falsity of statement p. On the right side, under the heading $\bar{p}$, we list the resulting truth or falsity for statement $\bar{p}$.

We can make this table simpler to read by replacing the words true and false by the symbols 1 and 0. A statement will be said to have truth value 1 if it is a true statement, and truth value 0 if it is false. Thus the truth table for $\bar{p}$ becomes:

p	$\bar{p}$
1	0
0	1

2-3 Truth Tables for $p \wedge q$ and for $p \vee q$

Suppose we are given the following two statements.

p: The patient has hepatitis.
q: The patient has anemia.

Consider the statement $p \wedge q$. In words, $p \wedge q$ means, "The patient has hepatitis and anemia." Thus $p \wedge q$ is true if both p and q are true and is false if p is false, or q is false, or both p and q are false. The truth table for $p \wedge q$ looks like this (see following page).

This table is also easy to interpret. For convenience, we are using a double vertical line to separate the table into two parts. On the left side of the table we have listed all possible combinations of truth values for statements p and q. On the right side of the table we have listed the resulting truth values for statement $p \wedge q$.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

We can think of the statement $p \wedge q$ as a "function machine" which acts on the truth vector [p, q]. Here [p, q] is one of the vectors [1, 1], [1, 0], [0, 1] or [0, 0], depending on the truth values of p and q, and the function $p \wedge q$ takes on the values 0 and 1, depending on the truth vector.

Now consider the statement $p \vee q$. In words, this means, "The patient has hepatitis or anemia (or both)." Then $p \vee q$ is true if either p or q is true (or both), and is false if p and q are both false. The truth table for $p \vee q$ is shown at right.

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

On the left, we list all the basic statements which appear in our example, together with all posible combinations of their truth values. On the right we list the resulting truth values for $p \vee q$.

Notice that we can change the order of the horizontal rows in a truth table without changing its meaning. Both tables at right contain the same information, and we can consider them both to be "the" truth table for the statement $\overline{p}$. There is always more than one correct way to display the information in a given truth table.

p	$\overline{p}$
1	0
0	1

p	$\overline{p}$
0	1
1	0

2-4 Building Large Tables Out of Small Ones

We can use the basic truth tables for $\overline{p}$, $p \wedge q$ and $p \vee q$ to construct the truth tables for more complicated statements. For example, the statement $\overline{\overline{p}}$, in words, means, "It is not true that the patient does not have hepatitis." It may not seem useful to consider such a cumbersome statement, but similar ones will be encountered in the study of circuits later. Therefore we will construct a truth table for $\overline{\overline{p}}$ now. We follow the steps below.

1. Starting with the statement p we can obtain $\overline{\overline{p}}$ by applying the "not" operation twice.

$$p \xrightarrow{\text{not}} \overline{p} \xrightarrow{\text{not}} \overline{\overline{p}}$$

Therefore we prepare a table showing all three statements p, $\overline{p}$ and $\overline{\overline{p}}$ (STEP 1 below).

2. We use the truth table for $\overline{p}$ on page 2 to fill in the first two columns (STEP 2 below).

3. To fill in the last column we note the $\bar{\bar{p}}$ is true when $\bar{p}$ is false and false when $\bar{p}$ is true. STEP 3 shows the final result.

STEP 1

p	$\bar{p}$	$\bar{\bar{p}}$

STEP 2

p	$\bar{p}$	$\bar{\bar{p}}$
1	0	
0	1	

STEP 3

p	$\bar{p}$	$\bar{\bar{p}}$
1	0	1
0	1	0

As another example, we will construct the truth table for $p \wedge \bar{q}$. The basic statements in $p \wedge \bar{q}$ are p and q, so we begin by filling in the first two columns with all the possible combinations of truth values for p and q. This gives STEP 1 below. STEP 2 involves remembering that each truth value for $\bar{q}$ will be the opposite of the truth value for q. Finally, we complete the last column by remembering that $p \wedge \bar{q}$ is true only when both p and $\bar{q}$ are true. STEP 3 is the result.

STEP 1

p	q	$\bar{q}$	$p \wedge \bar{q}$
1	1		
1	0		
0	1		
0	0		

STEP 2

p	q	$\bar{q}$	$p \wedge \bar{q}$
1	1	0	
1	0	[illegible]	
0	1	0	
0	0	1	

STEP 3

p	q	$\bar{q}$	$p \wedge \bar{q}$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

To find the truth table for $\overline{p \wedge q}$ we begin as usual. The third column is headed $p \wedge q$, because we will need the values for $p \wedge q$ to find the values for $\overline{p \wedge q}$ (STEP 1 below). We can find the values for $p \wedge q$ either by looking back at the table we developed earlier, or by merely remembering that $p \wedge q$ is true only when p and q are both true (see STEP 2). Finally, the column for $\overline{p \wedge q}$ will have the values that are opposite those for $p \wedge q$ (see STEP 3).

STEP 1

p	q	$p \wedge q$	$\overline{p \wedge q}$
1	1		
1	0		
0	1		
0	0		

STEP 2

p	q	$p \wedge q$	$\overline{p \wedge q}$
1	1	1	
1	0	0	
0	1	0	
0	0	0	

STEP 3

p	q	$p \wedge q$	$\overline{p \wedge q}$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

Once again note that we have split the table into two parts with a double vertical line. On the left we have all the basic statements and all combinations of their truth values. On the right we have the given statement $\overline{p \wedge q}$, together with the simpler statement $p \wedge q$, which it contains. The body of the table lists the truth values of these statements resulting from the various combinations of truth values for p and q.

2-5 Truth Tables Using Three Basic Statements

These same methods work for statements built up from more than two basic statements. For example, if our basic statements are p, q and r, we can construct the truth table for $p \wedge q \wedge r$.

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STEP 1

p	q	r	$p \wedge q$	$p \wedge q \wedge r$
1	1	1		
1	0	1		
0	1	1		
0	0	1		
1	1	0		
1	0	0		
0	1	0		
0	0	0		

STEP 2

p	q	r	$p \wedge q$	$p \wedge q \wedge r$
1	1	1	1	
1	0	1	0	
0	1	1	0	
0	0	1	0	
1	1	0	1	
1	0	0	0	
0	1	0	0	
0	0	0	0	

Notice that we have to be careful to list all possible combinations of the truth values for p, q and r. For three basic statements there are eight possibilities. We filled in the third column by remembering that $p \wedge q$ is true only when both p and q are true.

Since $p \wedge q \wedge r$ is true only when both $p \wedge q$ and r are true, we can now complete the last column.

We could just as well have constructed the table by finding the truth values of $q \wedge r$ instead of $p \wedge q$ in the fourth column. The right hand column would still be the same because $p \wedge (q \wedge r)$ is the same statement as $(p \wedge q) \wedge r$.

STEP 3

p	q	r	$p \wedge q$	$p \wedge q \wedge r$
1	1	1	1	1
1	0	1	0	0
0	1	1	0	0
0	0	1	0	0
1	1	0	1	0
1	0	0	0	0
0	1	0	0	0
0	0	0	0	0

The table above (STEP 3) can be summarized in the following shorter version.

p	q	r	$p \wedge q \wedge r$
1	1	1	1
all other combinations			0

The table at right (Short Version given below) is a truth table for $p \vee q \vee r$. Go over each column to make sure that you understand it. Remember that $p \vee q$ is <u>false</u> only when p and q are both <u>false</u>. (Look back at the table for $p \vee q$ that was developed earlier.)

p	q	r	$p \vee q$	$p \vee q \vee r$
1	1	1	1	1
1	0	1	1	1
0	1	1	1	1
0	0	1	0	1
1	1	0	1	1
1	0	0	1	1
0	1	0	1	1
0	0	0	0	0

SHORT VERSION

p	q	r	$p \vee q \vee r$
0	0	0	0
all other combinations			1

As a last example, consider the statement $p \wedge (q \vee \overline{r})$. We construct a truth table for this statement by first noting that $p \wedge (q \vee \overline{r})$ depends on p and $(q \vee \overline{r})$, and that $q \vee \overline{r}$ depends on q and $\overline{r}$.

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p	q	r	$\bar{r}$	$q \vee \bar{r}$	$p \wedge (q \vee \bar{r})$
1	1	1	0	1	1
1	0	1	0	0	0
0	1	1	0	1	0
0	0	1	0	0	0
1	1	0	1	1	1
1	0	0	1	1	1
0	1	0	1	1	0
0	0	0	1	1	0

SHORT VERSION

p	q	r	$p \wedge (q \vee \bar{r})$
1	1	1	1
1	1	0	1
1	0	0	1
all other combinations			0

Study once again the truth tables we have constructed in this section. Remember that the left side of a truth table contains all the possible combinations of truth values of the basic statements p and q (or p, q, r, etc.). Also, the tables build up to more complicated expressions a step at a time, from left to right. When parentheses are present, the expression in parentheses is treated before the whole expression.

PROBLEM SET 2:

Suppose the following two statements are given.

p: The patient has a fever.

q: The patient has normal reflexes.

Then $p \wedge q$ is the following statement.

The patient has a fever and normal reflexes.

Supply the word <u>true</u> or <u>false</u> for each blank.

1. Suppose both p and q are true. The $p \wedge q$ is __?__.
2. Suppose p is true and q is false. Then $p \wedge q$ is __?__.
3. Suppose p is false and q is true. Then $p \wedge q$ is __?__.
4. Suppose both p and q are false. Then $p \wedge q$ is __?__.

If p and q are the same statements as above, then $p \wedge \bar{q}$ is the following statement.

The patient has a fever and abnormal reflexes.

Supply the word <u>true</u> or <u>false</u> for each blank.

5. If p and q are both true, then $p \wedge \bar{q}$ is __?__.
6. If p is true and q is false, then $p \wedge \bar{q}$ is __?__.
7. If p is false and q is true, then $p \wedge \bar{q}$ is __?__.
8. If both p and q are false, then $p \wedge \bar{q}$ is __?__.

Suppose p, q, r are three statements. Supply the word <u>true</u> or <u>false</u> for each blank.

9. The statement $p \wedge q$ is true only when both statements p and q are __?__.
10. The statement $p \vee q$ is false only when p is __?__ and q is __?__.
11. The statement $p \wedge \bar{q}$ is true only when p is __?__ and $\bar{q}$ is __?__.
12. The statement $p \wedge q \wedge r$ is true only when p, q and r are all __?__.
13. The statement $p \vee q \vee r$ is false only when p is __?__, q is __?__ and r is __?__.

Complete the following truth tables.

14. Truth table for $\overline{p} \wedge q$.

p	q	$\overline{p}$	$\overline{p} \wedge q$
1	1	0	
1	0	0	
0	1	1	
0	0	1	

15. Truth table for $\overline{p} \vee \overline{q}$.

p	q	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$
1	1		0	
1	0		1	
0	1		0	
0	0		1	

16. Truth table for $\overline{p \wedge q}$.

p	q	$p \wedge q$	$\overline{p \wedge q}$
1	1		
1	0		
0	1		
0	0		

17. Truth table for $p \vee (q \wedge r)$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
1	1	1		
1	0	1		
0	1	1		
0	0	1		
1	1	0		
1	0	0		
0	1	0		
0	0	0		

18. Construct a truth table for $p \wedge \overline{q}$.
19. Construct a truth table for $\overline{p} \vee q$.
20. Construct a truth table for $\overline{\overline{p} \vee q}$.

*21. In this problem you will see how a type of truth table can be used to solve a logical puzzle. Start by reading the problem.

Brown, Clark, Jones and Smith are four doctors, one a brain surgeon, one a psychiatrist, one a dermatologist and one an acupuncturist (although not necessarily in that order). Use the information below to decide each person's specialty.

a. Brown is not the dermatologist.
b. Brown and Jones observed the brain surgeon's last operation.
c. Both Smith and the acupuncturist have been treated by the dermatologist.
d. The acupuncturist has treated both Clark and Brown.

In order to solve the problem, we start with the table below which has the names of the doctors along one side and the specialties along the other. Then we put a zero in each square that we can eliminate and a 1 in the proper square when we match a name with a specialty. For example, the sentence "Brown is not the dermatologist" allows us to put a zero in the position shown below. Your job is to complete the table.

	Clark	Brown	Jones	Smith
Acupuncturist				
Brain Surgeon				
Dermatologist		0		
Psychiatrist				

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*22. In a certain clinic the positions of surgeon, radiologist and anesthesiologist are held by Dr. Miasma, Dr. Roentgen and Dr. Stitch, although not necessarily in that order. Use the following information to decide which position each woman fills.

a. The anesthesiologist, who was an only child, earns the least.

b. Dr. Stitch, who married Roentgen's brother, earns more than the radiologist.

*23. The five beds in a certain hospital room are occupied by the following five persons: Ascot, Burke, Fosdick, Twang and Snerd. Use the information below to decide who occupies each bed.

a. The patients in Beds 2 and 4 will be operated on in the morning, and the others in the afternoon.

b. Both Ascot and Fosdick are married.

c. Ascot and Burke will be operated on at the same time.

d. Twang's operation is scheduled for the morning and Snerd's is scheduled for the afternoon.

e. The persons in Beds 2 and 3 are single and live together.

f. The person in Bed 5 is neither Twang nor Snerd.

g. Twang and Snerd live across the street from each other.

*24. The members of a baseball team are Andy, Ed, Harry, Paul, Allen, Sam, Bill, Jerry and Mike. Andy dislikes the catcher. Ed's sister is engaged to the second baseman. The center fielder is taller than the right fielder. Harry and the third baseman live in the same building. Paul and Allen each won \$20 from the pitcher at pinochle. Ed and the outfielders play poker during their free time. The pitcher's wife is the third baseman's sister. All the <u>battery</u> (pitcher and catcher) <u>and infield</u>, except Allen, Harry and Andy, are shorter than Sam. Paul, Andy and the shortstop lost \$50 each at the racetrack. Paul, Harry, Bill and the catcher took a trouncing from the second baseman at pool. Sam is undergoing a divorce suit. The catcher's wife and the third baseman's wife both have red hair. Ed, Paul, Jerry, the right fielder and the center fielder are bachelors. The others are married. The shortstop, the third baseman and Bill each cleaned up \$100 betting on the fight. One of the outfielders is either Mike or Andy. Jerry is taller than Bill. Mike is shorter than Bill. Both Mike and Bill are heavier than the third baseman. With these facts determine the names of the men playing the various positions on the baseball team.

SECTION 3: EQUIVALENT STATEMENTS

3-1 <u>The Idea of Equivalence</u>

In the language of algebra, we often encounter expressions which agree in value whenever the same numbers are substituted into both. For example, the two expressions $x(3 - y)$ and $3x - xy$ will agree for any values of x and y chosen. If $x = 4$ and $y = 1$ they both are equal to 8, and so on. When two expressions always agree we say they are equal. We write

$$x(3 - y) = 3x - xy$$

In the language of logical statements, there is a parallel situation. Often we encounter statements which always agree in truth value, that is, they "say the same thing." Rather than saying that the two statements are equal, we say they are equivalent. The idea of equivalence will be explored in this section.

3-2 Truth Tables and the Meaning of Statements

Notice that a truth table does not depend on the information contained in the statements involved; it depends only on how those basic statements are joined together by the operations (and parentheses). For example, the truth table for $p \wedge q$ will be the same, no matter what the basic statements p and q actually are. In fact, we can change the letters we use to denote the basic statements, and the right-hand column in the truth table will still look the same: $r \wedge s$ has the same truth table as $p \wedge q$.

Consider the following statements.

p: The patient has a broken leg.
q: The patient has a concussion.

What is the difference between the three statements q, $\bar{q}$ and $\bar{\bar{q}}$? In words, we have:

q: The patient has a concussion.
$\bar{q}$: The patient does not have a concussion.
$\bar{\bar{q}}$: It is not true that the patient does not have a concussion.

A close look at these statements suggests that $\bar{\bar{q}}$ merely says the same thing as q.

We can see this situation more clearly by looking at the truth table for $\bar{\bar{q}}$ (at right). This table shows the relation between the truth values of the statements q, $\bar{q}$ and $\bar{\bar{q}}$.

q	$\bar{q}$	$\bar{\bar{q}}$
1	0	1
0	1	0

Notice that the columns for q and $\bar{\bar{q}}$ are identical. This means that q and $\bar{\bar{q}}$ always have the same truth values--they are always both true or both false. In a sense, then, they "say the same thing."

Whenever two statements have the same truth-table columns--that is, they are either both true or both false in every case--we say the statements are equivalent. We use the equality sign to symbolize equivalence. Thus, by our definition, q and $\bar{\bar{q}}$ are equivalent, and we write

$$q = \bar{\bar{q}}$$

The above equivalence will be very important when you begin studying logic circuits.

3-3 More Examples of Equivalence

As another example consider the two statements $\overline{p \wedge q}$ and $\bar{p} \vee \bar{q}$. In words, we have:

$\overline{p \wedge q}$: "It is not true that the patient has both a broken leg and a concussion."

$\overline{p} \vee \overline{q}$: "Either the patient doesn't have a broken leg, or he doesn't have a concussion (or both)."

Since these two statements mean the same thing, we would expect them always to have the same truth values--to be equivalent. Are they really equivalent? Let us construct a truth table and see.

p	q	$\overline{p}$	$\overline{q}$	$p \wedge q$	$\overline{p \wedge q}$	$\overline{p} \vee \overline{q}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

The two columns for $\overline{p \wedge q}$ and $\overline{p} \vee \overline{q}$ are in fact the same. So these two statements are equivalent, by our definition. In symbols,

$$\overline{p \wedge q} = \overline{p} \vee \overline{q}$$

Similarly, consider the statements $\overline{p \vee q}$ and $\overline{p} \wedge \overline{q}$. The truth table is given at right. Here again, the last two columns are the same, so we have another equivalence, this time between $\overline{p \vee q}$ and $\overline{p} \wedge \overline{q}$.

p	q	$\overline{p}$	$\overline{q}$	$p \vee q$	$\overline{p \vee q}$	$\overline{p} \wedge \overline{q}$
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	0	1	1

At right is a list of equivalences we have found so far. The first equivalence is called "the law of the double negative"; the other two, together, are called "DeMorgan's laws." These equivalences are always true, no matter what the statements p and q represent. This is so because equivalences depend only on truth tables.

$$q = \overline{\overline{q}}$$
$$\overline{p \wedge q} = \overline{p} \vee \overline{q}$$
$$\overline{p \vee q} = \overline{p} \wedge \overline{q}$$

These three equivalences will be used as basic laws to transform complex statements into simpler statements which "say the same thing." When working on electrical circuits, these laws will enable you to find simpler circuits that will behave in the same manner as more complicated circuits. Because these three laws are so important, you should memorize them.

PROBLEM SET 3:

Suppose the following statements are given.

s: Horace has consumed 6 ounces of alcohol

t: Horace is unconscious.

Match each symbolic statement below with the sentence on the right which expresses its meaning.

1. $\overline{s}$
2. $\overline{t}$
3. $s \vee t$
4. $\overline{s \vee t}$
5. $\overline{s} \wedge \overline{t}$

A. Horace is not unconscious.

B. The following statement is not true: Horace has consumed 6 ounces of alcohol or Horace is unconscious.

C. Horace has not consumed 6 ounces of alcohol and Horace is not unconscious.

D. Horace has consumed 6 ounces of alcohol or Horace is unconscious.

E. Horace has not consumed 6 ounces of alcohol.

6. Which two symbolic statements above are different ways of saying the same thing? Express the equivalence as an equation.

7. a. Complete the truth table at right.

b. Which two statements turned out to be equivalent?

p	q	$\overline{q}$	$p \wedge q$	$p \wedge \overline{q}$	$(p \wedge q) \vee (p \wedge \overline{q})$
1	1				
1	0				
0	1				
0	0				

8. a. Complete the truth table at right.

b. Which two statements are equivalent?

r	s	$\overline{r}$	$r \vee s$	$\overline{r} \vee s$	$(r \vee s) \wedge (\overline{r} \vee s)$
1	1				
1	0				
0	1				
0	0				

9. a. Complete the truth table.

b. Which two statements are equivalent?

p	q	$\overline{p}$	$\overline{q}$	$\overline{p} \vee q$	$p \wedge \overline{q}$	$\overline{p \wedge \overline{q}}$
1	1					
1	0					
0	1					
0	0					

10. a. Complete the truth table.

b. Which two statements are equivalent?

s	t	$\overline{s}$	$\overline{t}$	$s \wedge \overline{t}$	$\overline{s} \vee t$	$\overline{\overline{s} \vee t}$
1	1					
1	0					
0	1					
0	0					

11. a. Complete the truth table.

b. Which two statements are equivalent?

s	t	$\overline{s}$	$\overline{t}$	$s \vee t$	$\overline{s} \wedge \overline{t}$	$\overline{\overline{s} \wedge \overline{t}}$
1	1					
1	0					
0	1					
0	0					

SECTION 4: SUBSTITUTION AND DE MORGAN'S LAWS

4-1 Substitution

In this section we will explore the algebra of statements. We will use De Morgan's laws to reduce complex statements to simpler equivalent ones. By substituting more complex statements for the basic statements p and q, we will be able to find other forms of De Morgan's laws, which will be easier to apply in particular problems.

Recall that De Morgan's laws are the equivalences

$$\overline{p \wedge q} = \bar{p} \vee \bar{q} \qquad\qquad \overline{p \vee q} = \bar{p} \wedge \bar{q}$$

where p and q are any two statements. Since p can be any statement, we can substitute some other statement, r for example, in place of p in the equivalences above. Then we get

$$\overline{r \wedge q} = \bar{r} \vee \bar{q} \qquad\qquad \overline{r \vee q} = \bar{r} \wedge \bar{q}$$

Likewise q can be any statement we wish. For example, q might be the statement $s \vee t$, in which case the equations above look like this.

$$\overline{r \wedge (s \vee t)} = \bar{r} \vee \overline{(s \vee t)} \qquad\qquad \overline{r \vee (s \vee t)} = \bar{r} \wedge \overline{(s \vee t)}$$

In the above process we started with two basic equivalences and then obtained two new ones by <u>substituting</u> new statements in place of p and q. In all substitution problems there are two basic rules you should keep in mind.

1. When a letter is to be replaced by substitution, it must be replaced <u>everywhere</u> it appears.

2. When the statement which is being substituted contains more than one letter, it should be enclosed by parentheses.

<u>EXAMPLE</u>:

Rewrite the equivalence $\bar{p} \wedge \bar{\bar{q}} = \bar{p} \wedge q$, when $p = \bar{s}$.

<u>SOLUTION</u>:

The letter to be replaced is p, which appears twice in the equation. In each place we substitute $\bar{s}$.

$$\bar{\bar{s}} \wedge \bar{\bar{q}} = \bar{\bar{s}} \wedge q$$

<u>EXAMPLE</u>:

Rewrite the equivalence $\overline{s \wedge t} = \bar{s} \vee \bar{t}$, when $s = \bar{p} \vee q$ and $t = \bar{q}$.

<u>SOLUTION</u>:

Carrying out the substitution in two steps, we begin by replacing s with $\bar{p} \vee q$. Since $\bar{p} \vee q$ involves more than one letter we include parentheses.

$$\overline{(\bar{p} \vee q) \wedge t} = \overline{(\bar{p} \vee q)} \vee \bar{t}$$

The second step of the substitution is the replacement of t by $\bar{q}$.

$$\overline{(\bar{p} \vee q) \wedge \bar{q}} = \overline{(\bar{p} \vee q)} \vee \bar{\bar{q}}$$

4-2 Other Properties of Equivalence

You may have already noticed that two statements which are equivalent to a third statement are equivalent to each other. This is true because two statements which always have the same truth values as a third statement must always have the same truth values as each other.

The number system has the same property. We know that

$$3 + 2 = 5$$

and $$6 - 1 = 5$$

It then follows that

$$3 + 2 = 6 - 1$$

The equivalent sign acts like the equal sign in another way. Consider the number

$$817 - (3 + 2)$$

We know that

$$3 + 2 = 5$$

So we must have

$$817 - (3 + 2) = 817 - 5$$

Similarly, we know

$$q = \bar{\bar{q}}$$

So if we take any statement in which q occurs and replace q by $\bar{\bar{q}}$, we get an equivalent statement. For example,

$$p \wedge q = p \wedge \bar{\bar{q}}$$

As another example, we know

$$\overline{p \vee q} = \bar{p} \wedge \bar{q} \quad \text{(De Morgan law)}$$

If we taken $\overline{r \wedge \overline{p \vee q}}$ and replace $\overline{p \vee q}$ by $\bar{p} \wedge \bar{q}$ we get $\overline{r \wedge \bar{p} \wedge \bar{q}}$, so we can conclude

$$\overline{r \wedge \overline{p \vee q}} = \overline{r \wedge \bar{p} \wedge \bar{q}}$$

4-3 <u>New Forms of De Morgan's Laws</u>

Let us recall one of De Morgan's laws.

$$\overline{s \wedge t} = \bar{s} \vee \bar{t}$$

In words, this law tells us that when two statements are joined by an "and" and topped by a "not" it means the same thing as joining the negatives of the statements by an "or." Suppose we start with the statement

$$\overline{p \wedge q \wedge r}$$

We can isolate the $p \wedge q$ part of the statement by parentheses (since only one connecting symbol $\wedge$ appears, we can insert parentheses wherever we wish).

$$\overline{(p \wedge q) \wedge r}$$

We now have two statements, $p \wedge q$ and r, joined by an "and" and topped by a "not." By De Morgan's law we can apply the "not" operation to each of the two statements separately and join them by an "or."

$$\overline{(p \wedge q) \wedge r} = \overline{(p \wedge q)} \vee \bar{r}$$

Finally De Morgan's law tells us that $\overline{p \wedge q} = \bar{p} \vee \bar{q}$. Substituting this on the right side gives

$$\overline{(p \wedge q) \wedge r} = (\bar{p} \vee \bar{q}) \vee \bar{r}$$

On each side only one connecting symbol appears, so we can eliminate parentheses, obtaining

$$\overline{p \wedge q \wedge r} = \bar{p} \vee \bar{q} \vee \bar{r}$$

This equation is a statement of <u>De Morgan's law for three statements</u>. We can use the technique for four statements.

$$\overline{p\wedge q\wedge r\wedge s} = \overline{(p\wedge q)\wedge(r\wedge s)}$$

$$= \overline{(p\wedge q)}\vee\overline{(r\wedge s)}$$

$$= \overline{p}\vee\overline{q}\vee\overline{r}\vee\overline{s}$$

So now we have three versions of the same De Morgan law.

$$\overline{p\wedge q} = \overline{p}\vee\overline{q}$$

$$\overline{p\wedge q\wedge r} = \overline{p}\vee\overline{q}\vee\overline{r}$$

$$\overline{p\wedge q\wedge r\wedge s} = \overline{p}\vee\overline{q}\vee\overline{r}\vee\overline{s}$$

By now you should see the pattern. We have a version of this law for any number of statements p, q, r, etc., and all these versions look much the same.

Suppose we had started with the other De Morgan law, $\overline{p\vee q} = \overline{p}\wedge\overline{q}$ instead of $\overline{p\wedge q} = \overline{p}\vee\overline{q}$. Then a series of similar steps would give us

$$\overline{p\vee q} = \overline{p}\wedge\overline{q}$$

$$\overline{p\vee q\vee r} = \overline{p}\wedge\overline{q}\wedge\overline{r}$$

$$\overline{p\vee q\vee r\vee s} = \overline{p}\wedge\overline{q}\wedge\overline{r}\wedge\overline{s}$$

and so forth.

4-4 Simplifying Statements

In the equivalence $\overline{p\vee q\vee r\vee s} = \overline{p}\wedge\overline{q}\wedge\overline{r}\wedge\overline{s}$, the statement on the right-hand side is simpler than the statement on the left-hand side in this way: on the left-hand side, the symbol ‾ applies to the whole complex statement $p\vee q\vee r\vee s$, while on the right-hand side the symbols ‾ are each applied to only a single letter p, q, r, s.

When we talk about <u>simplifying</u> a statement, we mean to change the statement into an equivalent statement with these two properties..

1. In the new statement, the bars denoting the "not" operation are restricted to single letters, p, q, r, etc.

2. In the new statement, no letter has more than one ‾ over it.

When we <u>simplify</u> a statement, do we really make it simpler? It's a matter of opinion. You might think $\overline{p\wedge q\wedge r\wedge s}$ is a simpler statement than $\overline{p}\vee\overline{q}\vee\overline{r}\vee\overline{s}$ because, for example, the first statement uses fewer symbols than the second. When you begin to study circuits, you will see that the circuits corresponding to a simplified statement are often easier to construct. It is also easier to see what will be needed for their construction.

Sometimes the phrase <u>reduce to standard form</u> is used instead of the word <u>simplify</u>. We will use the word <u>simplify</u>, because it's simpler.

Some statements to simplify:

EXAMPLE 1: Simplify $\overline{p\wedge\overline{q}}$.

STEP 1: $\overline{p\wedge\overline{q}} = \overline{p}\vee\overline{\overline{q}}$ De Morgan

STEP 2: $= \overline{p}\vee q$ Because $q = \overline{\overline{q}}$

Here we get Step 1 in this way.

STEP $\frac{1}{2}$: $\overline{p\wedge r} = \overline{p}\vee\overline{r}$ De Morgan

STEP 1: $\overline{p\wedge\overline{q}} = \overline{p}\vee\overline{\overline{q}}$ Replacing r by $\overline{q}$ in Step $\frac{1}{2}$

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EXAMPLE 2: Simplify $\overline{\overline{p} \vee (q \wedge s)}$

STEP 1: $\overline{\overline{p} \vee (q \wedge s)} = \overline{\overline{p}} \wedge \overline{(q \wedge s)}$ De Morgan

STEP 2: $= p \wedge (\overline{q} \vee \overline{s})$ De Morgan

In more detail, this is what we have done.

STEP $\frac{1}{2}$: $\overline{t \vee r} = \overline{t} \wedge \overline{r}$ De Morgan

STEP 1: $\overline{\overline{p} \vee (q \wedge s)} = \overline{\overline{p}} \wedge \overline{(q \wedge s)}$ Replacing t by $\overline{p}$ and r by $q \wedge s$ in Step $\frac{1}{2}$

STEP $1\frac{1}{3}$: $= p \wedge \overline{(q \wedge s)}$ Because $p = \overline{\overline{p}}$

STEP $1\frac{2}{3}$: $\overline{q \wedge s} = \overline{q} \vee \overline{s}$ De Morgan

STEP 2: $\overline{\overline{p} \vee (q \wedge s)} = p \wedge (\overline{q} \vee \overline{s})$ Replacing $\overline{q \wedge s}$ by $\overline{q} \vee \overline{s}$ in Step $1\frac{1}{3}$

PROBLEM SET 4:

1. (Multiple Choice) Which of the following represents the equivalence $\overline{p \wedge q} = \overline{p} \vee \overline{q}$ when r is substituted for p?

a. $\overline{p \wedge r} = \overline{p} \vee \overline{q}$ b. $\overline{r \wedge q} = \overline{r} \vee \overline{q}$ c. $\overline{p \wedge q} = \overline{r} \vee \overline{q}$ d. $\overline{p \vee r} = \overline{p} \vee \overline{r}$

2. (Multiple Choice) Which of the following represents the equivalence $p \vee \overline{\overline{q}} = p \vee q$ when s is substituted for p?

a. $p \vee \overline{\overline{s}} = p \vee s$ b. $p \vee \overline{\overline{q}} = s \vee q$ c. $s \vee \overline{\overline{q}} = s \vee q$

3. (Multiple Choice) Which of the following represents the equivalence $\overline{s \vee t} = \overline{s} \wedge \overline{t}$ when $u \wedge \overline{w}$ is substituted for t?

a. $\overline{s \vee t} = \overline{s} \wedge \overline{(u \wedge \overline{w})}$ b. $\overline{s \vee (u \wedge \overline{w})} = \overline{s} \wedge \overline{(u \wedge \overline{w})}$ c. $s \vee \overline{(u \wedge \overline{w})} = \overline{s} \wedge \overline{(u \wedge \overline{w})}$

Rewrite each statement, making the indicated substitutions.

4. $\overline{p \vee q} = \overline{p} \wedge \overline{q}$; $p = s$

5. $\overline{p \wedge q} = \overline{p} \vee \overline{q}$; $q = \overline{w}$

6. $s \wedge \overline{\overline{t}} = s \wedge t$; $s = p \vee q$

7. $\overline{w \vee u} = \overline{w} \wedge \overline{u}$; $u = r \wedge s$

8. $\overline{p \vee q} = \overline{p} \wedge \overline{q}$; $p = r \wedge s$, $q = \overline{s}$

9. $\overline{p \wedge q} = \overline{p} \vee \overline{q}$; $p = \overline{r}$, $q = r \vee \overline{s}$

Simplify the following statements.

10. $\overline{\overline{s}}$

11. $\overline{r \wedge \overline{s}}$

12. $\overline{\overline{p} \wedge r}$

13. $\overline{\overline{s} \wedge \overline{t}}$

14. $\overline{\overline{q} \vee r}$

15. $\overline{\overline{p} \vee \overline{q}}$

16. $\overline{p \vee \overline{r}}$

17. $\overline{p \vee q \vee r}$

Show steps which justify each of the following.

18. $\overline{\overline{p} \wedge \overline{q} \wedge \overline{r}} = p \vee q \vee r$

19. $\overline{\overline{(q \vee r)} \wedge \overline{s}} = q \vee r \vee s$

20. $\overline{\overline{(p \wedge q)} \vee \overline{(r \wedge s)}} = p \wedge q \wedge r \wedge s$

SECTION 5: INTRODUCTION TO CIRCUITS

5-1 Introduction

In Section 2 you saw how truth tables are used to decide the truth value of a given statement, when the truth values of the basic statements p, q, r, etc., are known. For example, the truth table of $p \vee \bar{q}$ is given at right.

p	q	$\bar{q}$	$p \vee \bar{q}$
1	1	0	1
1	0	1	1
0	1	0	0
0	0	1	1

The second row of this table tells us that if p has truth value 1 and q has truth value 0, then $\bar{q}$ has truth value 1 and $p \vee \bar{q}$ has truth value 1. In other words, if [p, q] = [1, 0] then $\bar{q}$ is 1 and $p \vee \bar{q}$ is 1. The third row tells us that if [p, q] = [0, 1] then $p \vee \bar{q}$ is 0.

By now you have seen how much work is required to construct truth tables for complicated statements. Fortunately, we can build electrical circuits which do most of the work of evaluating truth tables. These circuits contain logical components called gates. When a large number of such circuits are combined properly, the result is a computer. You will work extensively with gates and circuits in Science class. In Mathematics class, you will study the logic aspect of these gates, so that you can understand the circuits you see in Science class.

5-2 An Example of a Circuit

When Norbert comes home from work, two things may start to happen.

1. The telephone starts ringing, and high-pressure phone salesmen try to sell him carpets, drapes and raffle tickets.

2. The doorbell starts ringing, and high-pressure door-to-door salesmen try to sell him toothbrushes, vacuum cleaners and encyclopedias.

Norbert can handle the telephone salesman if the doorbell isn't ringing, and he can handle the door-to-door salesmen if the telephone isn't ringing. But when the telephone and the doorbell start ringing at the same time he gets migraine headaches.

Norbert can avoid getting a headache if he immediately stops what he's doing and meditates. He wants to build an electrical circuit which will turn on a flashing red light to remind him to meditate whenever the doorbell and telephone ring at the same time.

Let's analyze this problem using symbolic logic. Let p, q, and r stand for the following statements.

p: The telephone is ringing.

q: The doorbell is ringing.

r: The red light is on.

Now we make the table at right which describes the way we want our circuit to behave.

statement p (true/false)	statement q (true/false)	statement r (true/false)
true	true	true
true	false	false
false	true	false
false	false	false

Suppose that we rewrite the table, using truth values of 0 and 1. The table is shown at right, next to the table for $p \wedge q$.

p	q	r
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

These two tables are the same, so we can say that r and $p \wedge q$ are equivalent statements. This is so, even though the statements p and q are not contained in r in any obvious way. If we were just given the statement r "out of the blue," we wouldn't know offhand that it depends in any way on statements p and q.

In fact, we know that r depends on p and q in the way that we have shown, because we have extra information which isn't stated specifically in statement r, but which is given in the discussion of the problem. We know that the light and the two bells must be wired to a device that will activate the light when both bells are ringing, and at no other time.

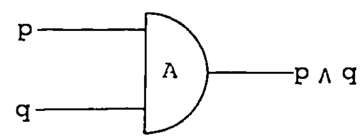
On the other hand, the two tables above differ in one important way. In the truth table of $p \wedge q$, the 1's and 0's stand for truth values. In the other table we can think of the 1's and 0's as indicating the presence or absence of an electrical current. In the p and q columns, 1 and 0 mean the presence or absence of the current which rings the telephone bell or doorbell. In the r column, 1 and 0 mean the presence or absence of the current which activates the light.

5-3 The AND Gate

Norbert doesn't have to waste his time designing the circuit to activate his light. Fortunately, there is a standard circuit component which behaves just the way Norbert wants his circuit to behave. This component is called an AND gate.

AND gates are easily available, already assembled and ready to use. Also, we don't need to bother about what's inside of an AND gate. We're only interested in what the AND gate does. So we can think of an AND gate as a little box with three wires--two to carry signals into the box and one to carry signals out. If there are any other wires, they only serve to supply power for the components inside the box. We won't worry about them, and we won't show them in our diagrams.

We use the symbol at right for an AND gate: Here are some points to remember about this symbol.



1. Both the shape of the symbol and the letter "A" tell us that this is the symbol for an AND gate, rather than one of the other gates you will see later.

2. The two lines p and q represent the inputs of the gate, the two wires which can carry electrical currents into the gate.

3. The line labeled $p \wedge q$ represents the output of the gate, the wire which may or may not carry a current out of the gate, depending on the inputs.

4. $p \wedge q$ is called the switching function of the gate. If we think of the gate as a function machine operating on [p, q], then $p \wedge q$ is the function the machine performs.

5. 1 designates current present, and 0 designates current absent.

Thus each of the inputs p and q may be 0 or 1, and the output $p \wedge q$ will be 0 or 1, depending on the inputs.

5-4 Other Logic Gates

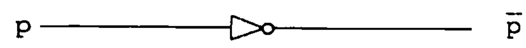
$p \wedge q$ is not the only statement we can form from the letters p and q. Some of the others are:

$$\overline{p} \qquad p \vee q \qquad \overline{p \wedge q} \qquad \overline{p \vee q}$$

There is a logic gate for each of these statements.

The first gate we will consider is the INVERT gate. This gate has only one input, instead of two. The switching function for the INVERT gate is $\overline{p}$. This means that the INVERT gate has output 1 only when the input is 0.

The symbol and table for the INVERT gate are given at right. The small circle, which appears in other gate symbols, stands for the "not" operation.



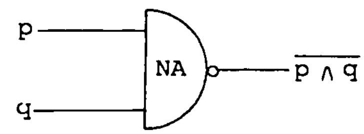
INVERT

INPUT	OUTPUT
p	$\overline{p}$
1	0
0	1

We now consider some other gates with two inputs and one output. The difference between them is in their switching functions.

The NAND gate has an output opposite to that of the AND gate (NAND stands for not AND). Hence the output is 1 only when at least one input is 0. The switching function is $\overline{p \wedge q}$ (not $p \wedge q$).

The symbol for the NAND gate is given at right. Notice the small circle which stands for "not."



By De Morgan's law we know that the switching function $\overline{p \wedge q}$ is equivalent to $\overline{p} \vee \overline{q}$. This fact will often be useful.

The table at right shows how the outputs of AND and NAND gates depend on their inputs. Notice that given the same inputs, the AND and NAND gates always have opposite outputs.

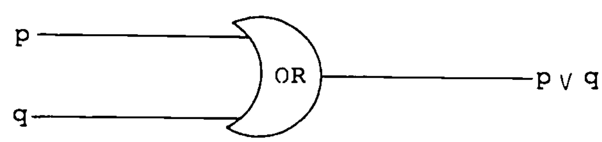
INPUTS		OUTPUTS	
		AND	NAND
p	q	$p \wedge q$	$\overline{p \wedge q}$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

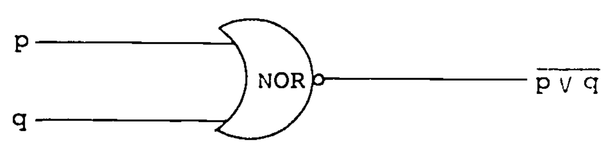
The statements $p \vee q$ and $\overline{p \vee q}$ give us switching functions for another pair of gates. These gates are related to each other in the same way as the AND and NAND gates.

The OR gate has output 1 when at least one of the inputs is 1. The NOR (not OR) gate has output 1 only when both inputs are 0. The table at right shows how the gates are related.

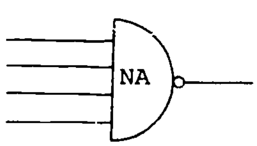
INPUTS		OUTPUTS	
		OR	NOR
p	q	$p \vee q$	$\overline{p \vee q}$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

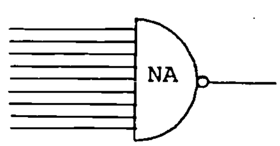
The symbols for these gates are given in the diagram at lower right. In this case also, the small circle in the NOR symbol stands for "not." By De Morgan's law, we can also write the switching function for the NOR gate in the equivalent form $\overline{p} \wedge \overline{q}$.





The gates we have presented are only the basic ones. There are many other types. In Science class you will often work with gates which have more than two inputs. The 4-input and 8-input NAND gates are shown below.

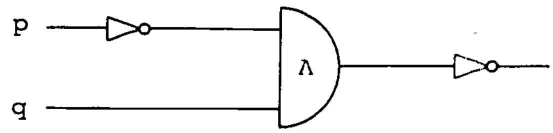




These gates follow the same logical pattern as the two-input NAND gate. That is, the output is 0 only when all the inputs are 1.

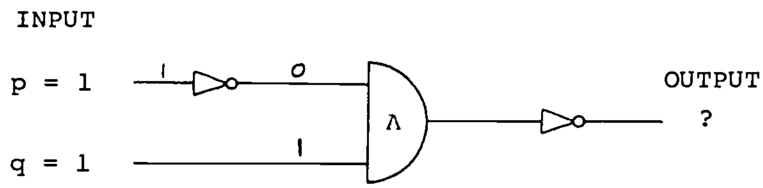
5-5 Circuits With More Than One Gate

Suppose we connect two INVERT gates and an AND gate in this way:

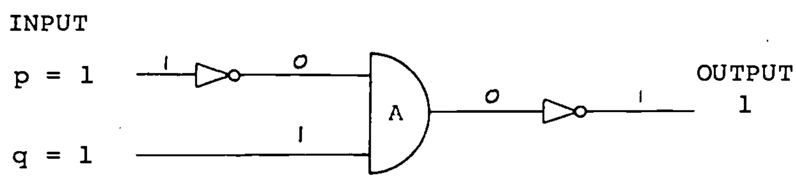


We can use the tables for the AND and INVERT gates to determine the behavior of this circuit.

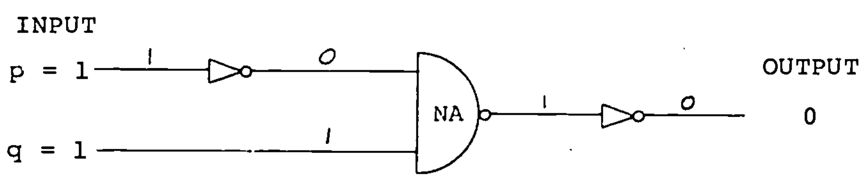
For example, if the input of the circuit is [p, q] = [1, 1], the first INVERT gate changes p = 1 to 0, so that the inputs of the AND gate are [0, 1]. We show this on the diagram in this way:



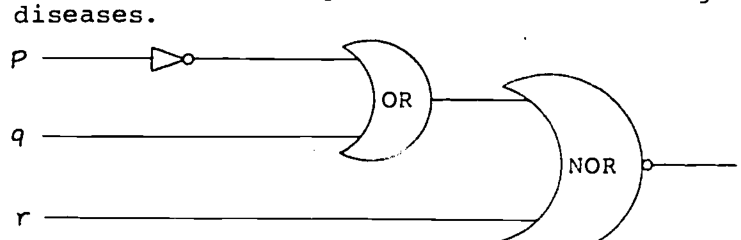
The input to the AND gate is [0, 1] and therefore the output is 0. The second INVERT gate then changes this to a 1.



If we had used a NAND gate instead of an AND gate, the result for [p, q] = [1, 1] would be:



Let's try a circuit with more inputs. You will be designing circuits such as these to diagnose diseases.



Suppose, for example, the input is [p, q, r] = [1, 0, 0]. Use the tables for the INVERT, OR and NOR gates to trace the behavior of the circuit, from left to right on the diagram.

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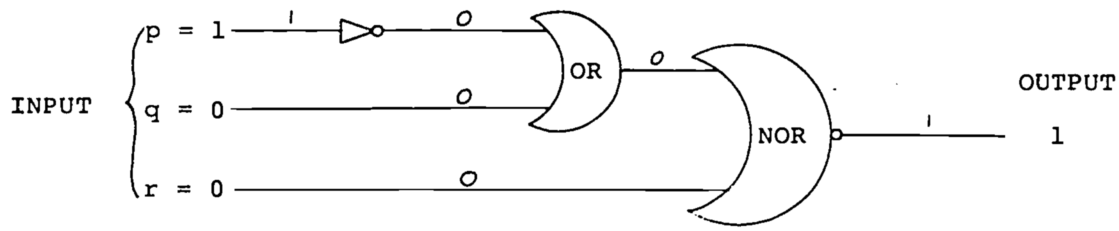
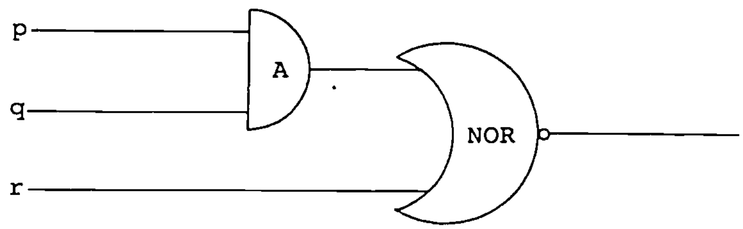
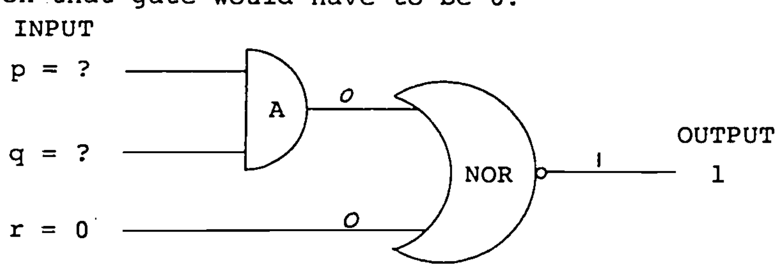


We can use these methods to find the output for each possible combination of inputs. If we arrange this information in the form of a table, we will have a truth table for the circuit.

We can also trace these diagrams backwards to find out what inputs are necessary to give a particular output. Consider this circuit.



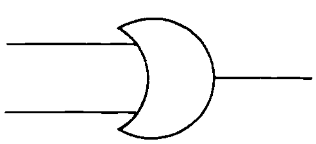
Which inputs give an output of 1? The table for the NOR gate (p. 23) tells us that both inputs on that gate would have to be 0:

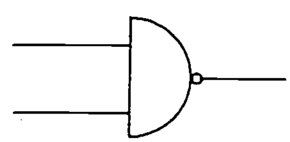


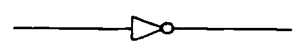
So r can only be 0, and the output of the AND gate must also be 0. This means that [p, q] can be [0, 0], [1, 0] or [0, 1], but not [1, 1]. So [p, q, r] can be [0, 0, 0], [1, 0, 0], or [0, 1, 0]. No other combinations give output 1 for this circuit.

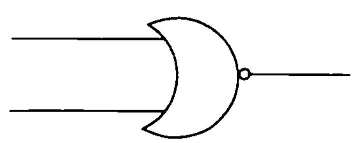
PROBLEM SET 5:

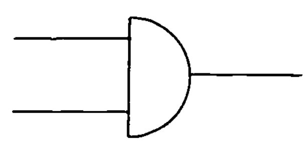
Write the name of each gate shown below.

1. 

2. 

3. 

4. 

5. 

6. Refer to the circuit on the right in this problem.

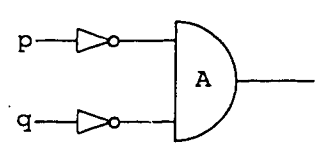
   a. If [p, q] = [1, 1] the output is ___?___.

   b. If [p, q] = [1, 0] the output is ___?___.

   c. If [p, q] = [0, 1] the output is ___?___.

   d. If [p, q] = [0, 0] the output is ___?___.

   e. This circuit has the same output as a(n) ___?___ gate.



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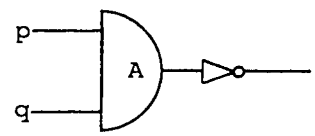
7. a. Complete the following truth table for the circuit on the right.

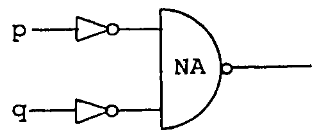
INPUTS		OUTPUT
p	q	
1	1	?
1	0	?
0	1	?
0	0	?



b. This circuit has the same output as a(n) __?__ gate.

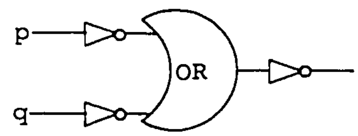
8. a. Construct an INPUT-OUTPUT truth table for the circuit on the right.

b. The circuit on the right has the same output as a(n) __?__ gate.

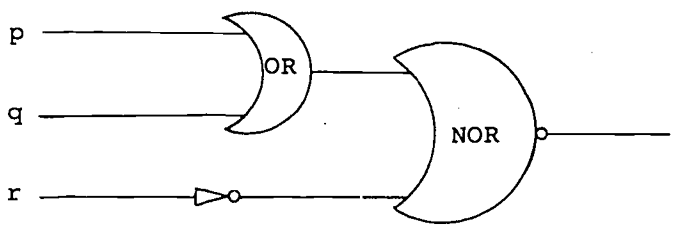


9. a. Construct a truth table for the circuit on the right.

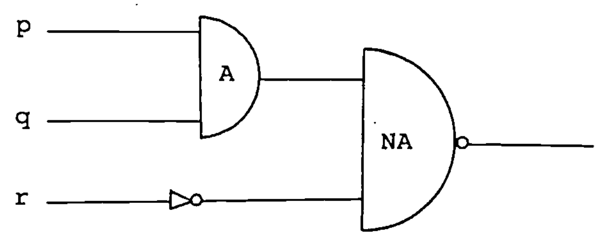
b. The circuit on the right has the same output as a(n) __?__ gate.



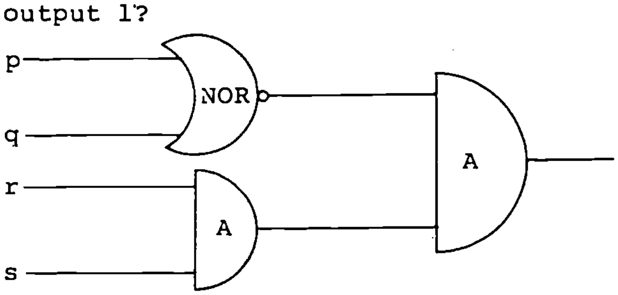
10. Write down all the input vectors [p, q, r] which will make the output 1.



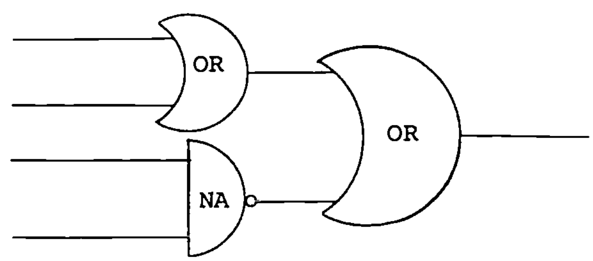
11. What input vectors [p, q, r] will make the output 0?



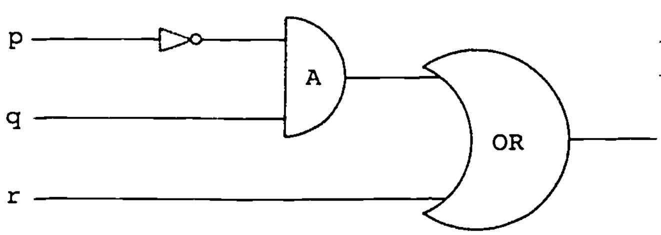
12. What input vectors will make the output 1?



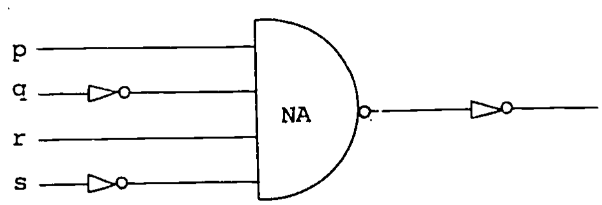
13. What input vectors will make the output 0?



14. Write down all the input vectors [p, q, r] which will make the output 1.



15. Write down all the input vectors [p, q, r, s] which will make the output 1.



SECTION 6: SWITCHING FUNCTIONS

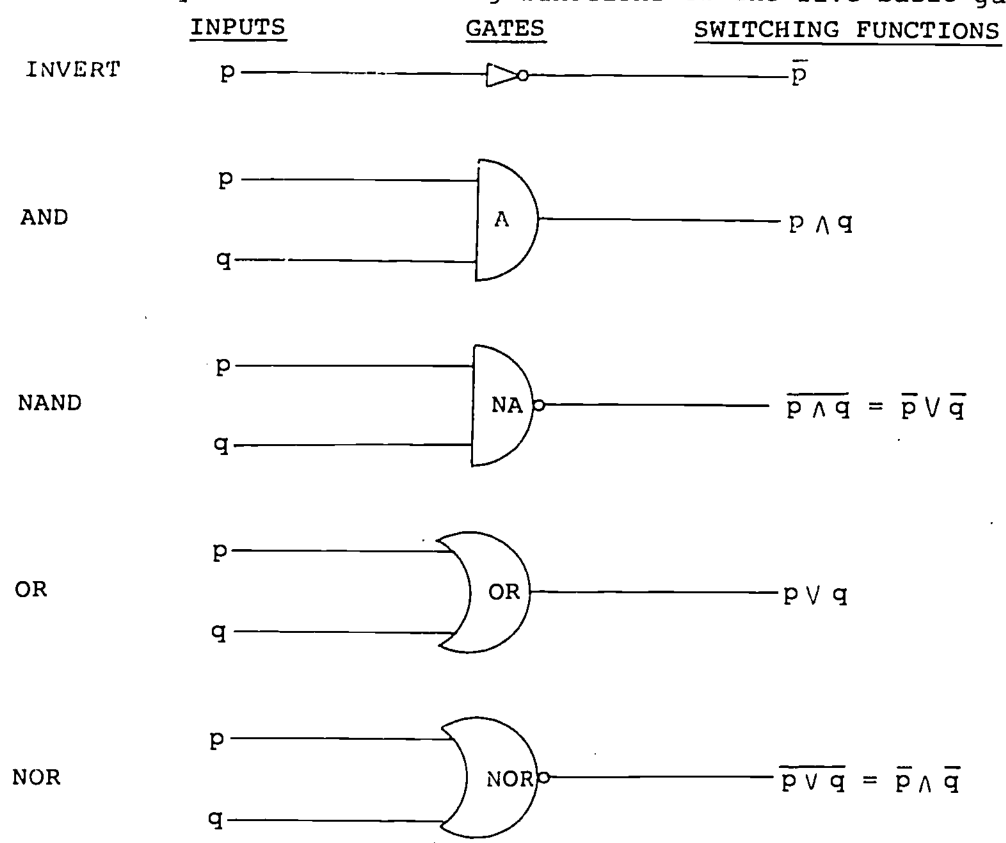
6-1 Introduction

In the last section, we described the AND, NAND, OR, NOR and INVERT gates. Each gate has a truth table which describes its outputs for various inputs. Each gate also has a switching function. The switching function is a logical statement with the same truth table as the circuit. When we want to describe what a gate does, it is much easier to write down the switching function than write out the whole truth table.

The same remarks go for more complicated circuits. A truth table can always be constructed by checking the output for each possible input combination. But it is more convenient to have a logical statement (switching function) to represent it. In this section you will see how to find switching functions for circuits containing several gates. Then you will see how to use switching functions to decide whether two circuits are equivalent (always have the same output).

6-2 Examples of Switching Functions

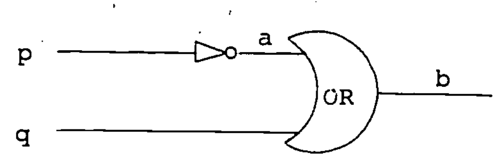
Here are the symbols and switching functions of the five basic gates.



Notice that we have written the switching functions for the NAND and NOR gates in two equivalent ways. We can do this because of De Morgan's laws.

$$\overline{p \wedge q} = \overline{p} \vee \overline{q} \qquad \overline{p \vee q} = \overline{p} \wedge \overline{q}$$

Now consider the slightly more complicated circuit at right. Look at the behavior of the circuit at the points labeled "a" and "b." At point a we have switching function $\overline{p}$, because this is the output of the INVERT gate. Then this output, together with q, are the inputs for the OR gate, to give us $\overline{p} \vee q$



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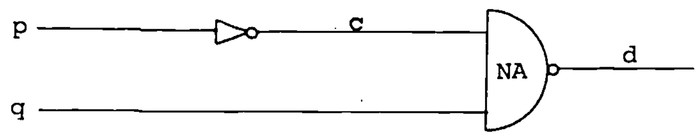
as the switching function at point b. Thus the switching function of the entire circuit is $\overline{p}\vee q$.

Similarly, consider the circuit at right. At point c we have $\overline{p}$. Therefore the inputs to the NAND gate are $\overline{p}$ and q. The effect of a NAND gate is to join the inputs by $\wedge$ (and) and apply $\overline{}$ (not) to the result. This gives $\overline{\overline{p}\wedge q}$ at point d. But we can simplify this by De Morgan's law.

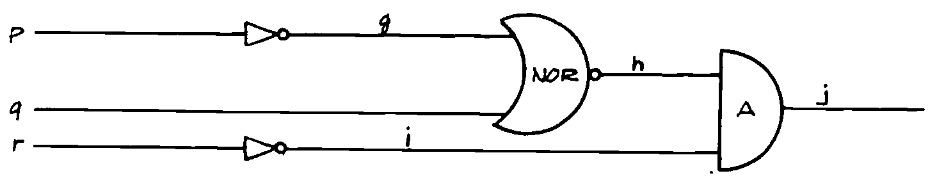


$$\overline{\overline{p}\wedge q} = \overline{\overline{p}}\vee\overline{q}$$

$$= p\vee\overline{q}$$

So the switching function for the entire circuit is $p\vee\overline{q}$.

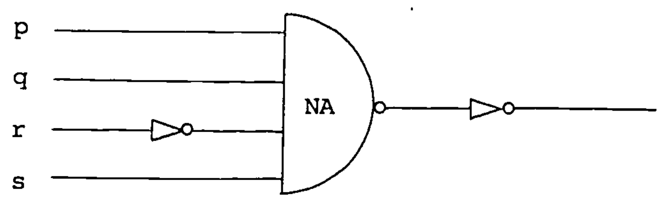
Now let's try something more complicated.



At point	switching function	reason
g	$\overline{p}$	output of INVERT
h	$\overline{\overline{p}\vee q} = p\wedge\overline{q}$	$\overline{p}$ and q are inputs of NOR; simplify using De Morgan's law.
i	$\overline{r}$	output of INVERT
j	$p\wedge\overline{q}\wedge\overline{r}$	h and i are inputs of AND

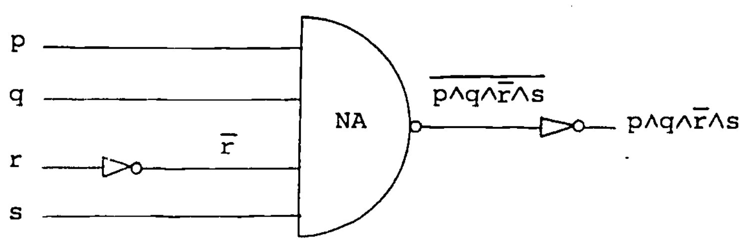
EXAMPLE:

Find the switching function of the entire circuit at right.



SOLUTION:

There is an INVERT gate on the r input wire so the four inputs to the NAND gate are p, q, $\overline{r}$ and s. Remembering that NAND means "not AND," the output from the NAND gate will be $\overline{p\wedge q\wedge \overline{r}\wedge s}$. The last INVERT gate then changes this to $\overline{\overline{p\wedge q\wedge \overline{r}\wedge s}}$, which simplifies to $p\wedge q\wedge \overline{r}\wedge s$. The completed diagram looks like this.



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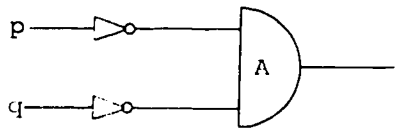
6-3 Equivalent Circuits

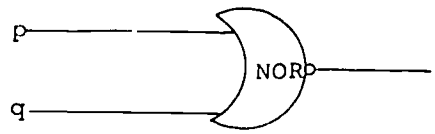
Recall that equivalent statements are ones which "say the same thing." Their truth values always agree. Similarly, two circuits which appear quite different may "do the same job." Their outputs may always agree. In that case, we say that the circuits are equivalent. The easiest way to tell whether two circuits are equivalent is to compare their switching functions. If the switching functions are equivalent statements, then the circuits are equivalent.

It is helpful to know when two circuits are equivalent, because often one circuit will be less costly and/or easier to build than the other. For instance, one circuit might use fewer gates, or fewer kinds of gates, or less expensive gates.

EXAMPLE:

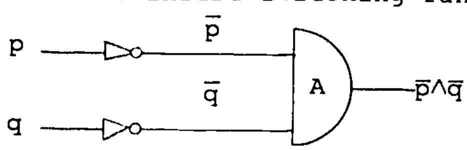
Determine whether the following two circuits are equivalent.

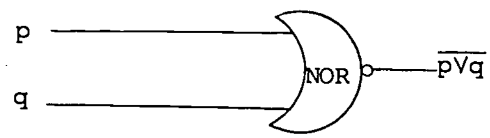




SOLUTION:

We first insert switching functions along both circuits.





We now compare the two switching functions representing the circuits, that is, $\bar{p}\wedge\bar{q}$ and $\overline{p\vee q}$. By De Morgan's law, these two statements are equivalent.

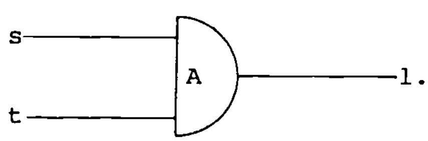
$$\bar{p}\wedge\bar{q} = \overline{p\vee q}$$

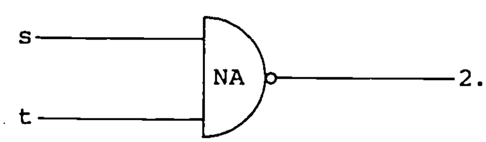
Therefore the two circuits are equivalent.

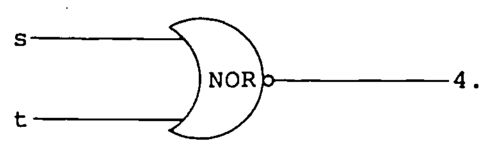
In fact, the circuit on the left appeared in Problem 6 of Problem Set 5. In that problem, the circuit turned out to have the same truth table as a NOR gate. In this example, we can see why that is true by looking at the switching functions of the circuits.

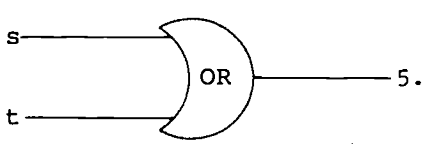
PROBLEM SET 6:

Give the switching functions corresponding to each number.

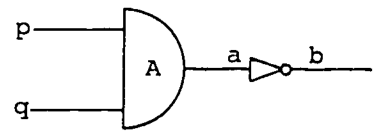








6. Refer to the circuit on the right.

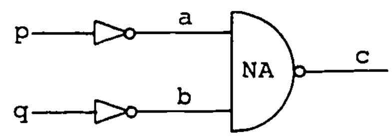


a. Switching Function a is ___(?)___.

b. Find Switching Function b and simplify using De Morgan's law.

c. Switching Function b is the same as that of a(n) ___(?)___ gate.

7. Refer to the circuit on the right.



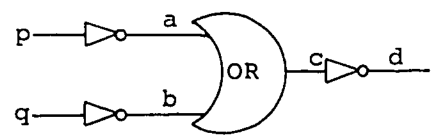
a. Switching Function a is ___(?)___.

b. Switching Function b is ___(?)___.

c. Switching Function c (simplified) is ___(?)___.

d. Switching Function c is the same as that of a(n) ___(?)___ gate.

8. Refer to the circuit on the right.



a. Switching Function a is ___(?)___.

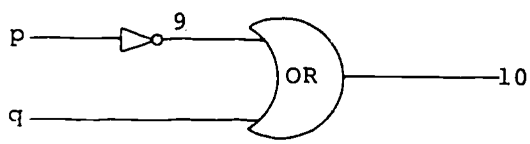
b. Switching Function b is ___(?)___.

c. Switching Function c is ___(?)___.

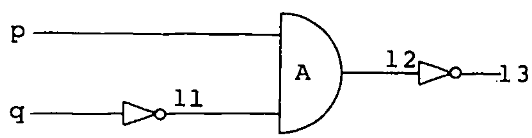
d. Switching Function d (simplified) is ___(?)___.

e. Switching Function d is the same as that of a(n) ___(?)___ gate.

Provide the numbered switching functions in the following circuits.



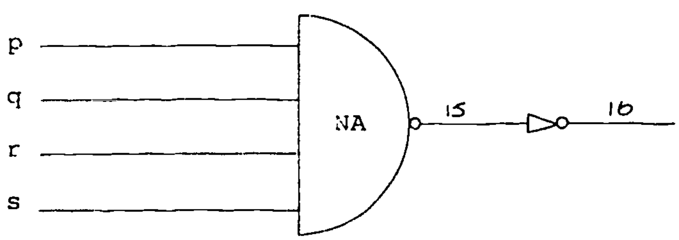
CIRCUIT A

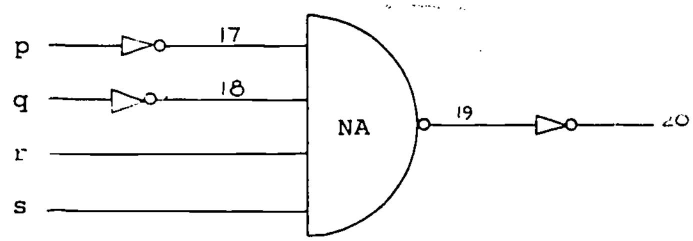


CIRCUIT B

14. Are Circuits A and B equivalent?

Provide the numbered switching functions in the following circuits.





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SECTION 7: TRUTH TABLES TO SWITCHING FUNCTIONS

7-1 Introduction

In the last two sections we have demonstrated how to find the truth table or the switching function of a given circuit. This section and the next will present the opposite problem--how to construct a circuit that has a given truth table or switching function. You will be using these techniques shortly in Science class. You will have a truth table or switching function describing a disease in terms of its symptoms, and you will design a circuit to diagnose the disease.

In this section you will see how to find the switching function that "goes with" a given truth table. In the next section you will see how to find the logic circuit that "goes with" a given switching function.

7-2 One Method of Finding Switching Functions

There are two slightly different methods of finding switching functions for a given truth table. In order to see how the first method works, consider the table at right below.

EXAMPLE 1:

Checking the right-hand column, you can see that the output is 1 only in row 2. In order to get that output, p must be 1 and q must be 0. Putting it another way, p must be 1 and $\overline{q}$ must be 1. Thus the output is 1 exactly when the statement $p \wedge \overline{q}$ is true. We can think of translating the row of inputs as follows.

INPUTS		OUTPUT
p	q	
1	1	0
1	0	1
0	1	0
0	0	0

$$\begin{array}{ccc} p \text{ is } 1 & \text{and} & q \text{ is } 0 \\ \downarrow & \downarrow & \downarrow \\ p & \wedge & \overline{q} \end{array}$$

The statement $p \wedge \overline{q}$ is the switching function corresponding to the table.

EXAMPLE 2:

Here, the second and third rows have output 1. From the previous example, we know that the statement corresponding to row 2 is

$$p \wedge \overline{q}$$

We now write a statement for row 3, using the input for that row.

INPUTS		OUTPUT
p	q	
1	1	0
1	0	1
0	1	1
0	0	0

$$\begin{array}{ccc} p \text{ is } 0 & \text{and} & q \text{ is } 1 \\ \downarrow & \downarrow & \downarrow \\ \overline{p} & \wedge & q \end{array}$$

Finally, we note that we get an output of 1 in the table if either the row 2 statement is true or the row 3 statement is true. The switching function for the table is therefore

$$(p \wedge \overline{q}) \vee (\overline{p} \wedge q)$$

In Section 1 you learned that there are two different kinds of or in logic. The inclusive or carries an implied or both. This is the or we designate by the symbol $\vee$.

The exclusive or carries an implied but not both. If you look closely at the table in Example 2, you will see that this is a truth table for exclusive or. That is, we get a 1 only when p is 1, or q is 1, but not both.

The method that we have been using to find switching functions for truth tables can be summarized as follows.

METHOD 1:

1. Find the rows of the truth table that have output 1.

2. Write a sentence for each row that tells which combinations of inputs give output 1.

3. Convert the sentence to symbols.

   a. Replace the word "and" by the symbol $\wedge$.

   b. Replace p is 1 by p and p is 0 by $\bar{p}$.

   c. Do the same for any other letters in the problem.

4. Join the statements from Step 3 by the symbol $\vee$ (or).

Here is an example with three inputs.

EXAMPLE 3:

The statement for the first row is $p \wedge \bar{q} \wedge r$, while the one for the second row is $\bar{p} \wedge \bar{q} \wedge \bar{r}$. Joining these two statements by the "or" symbol we get the switching function

$$(p \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge \bar{q} \wedge \bar{r})$$

INPUTS			OUTPUT
p	q	r	
1	0	1	1
0	0	0	1
all others			0

7-3 A Limitation of Method 1

In Examples 1 to 3, no more than half of the outputs were 1. If more than half of the outputs in a table are 1, then Method 1 tends to give unwieldy results. We will still get true switching functions, but we won't get the simplest possible equivalent form of the switching function. Example 4 below illustrates this.

EXAMPLE 4:

We want to find the switching function for the table on the far right. Ignore the table on the left for a minute. Method I gives the switching function

$$(p \wedge q) \vee (\bar{p} \wedge q) \vee (\bar{p} \wedge \bar{q})$$

This is correct, but there is a simpler equivalent version.

$$\bar{p} \vee q$$

INPUTS		OUTPUT
p	q	
1	1	0
1	0	1
0	1	0
0	0	0

INPUTS		OUTPUT
p	q	
1	1	1
1	0	0
0	1	1
0	0	1

We get this simpler version by comparing the two tables. Notice that:

1. The table on the left is the same as Example 1, and thus has switching function $p \wedge \bar{q}$, by Method 1.

2. The outputs in the right-hand table are always the opposite of the outputs in the left-hand table. This means that the switching function for the right-hand table must be $\overline{p \wedge \bar{q}}$.

3. This switching function can then be simplified using De Morgan's laws.

$$\overline{p \wedge \bar{q}} = \bar{p} \vee \bar{\bar{q}}$$
$$= \bar{p} \vee q$$

We can summarize these steps as another method.

METHOD 2:

1. Look at just those rows of the truth table that have output 0.

2. Find the switching function for each row with output 0, just as you did in Method 1 for rows with output 1.

3. Take the negation of the entire switching function.

Let's try some examples.

EXAMPLE 5:

INPUTS		OUTPUT
p	q	
1	1	0
1	0	1
0	1	1
0	0	0

Here, we consider only the first and last rows, where we have output 0. The statement for the first row is $p \wedge q$ while that for the last one is $\bar{p} \wedge \bar{q}$. Joining the two by "or" we get

$$(p \wedge q) \vee (\bar{p} \wedge \bar{q})$$

We then take the negation of the entire statement.

$$\overline{(p \wedge q) \vee (\bar{p} \wedge \bar{q})}$$

We simplify this, using De Morgan's laws.

$$\begin{aligned} \overline{(p \wedge q) \vee (\bar{p} \wedge \bar{q})} &= \overline{p \wedge q} \wedge \overline{\bar{p} \wedge \bar{q}} \\ &= (\bar{p} \vee \bar{q}) \wedge (\bar{\bar{p}} \vee \bar{\bar{q}}) \\ &= (\bar{p} \vee \bar{q}) \wedge (p \vee q) \end{aligned}$$

Example 5, like Example 2, is the truth table for exclusive or. So we have two equivalent translations of exclusive or into symbolic notation.

$(p \wedge \bar{q}) \vee (\bar{p} \wedge q)$--from Example 2, using Method 1

$(\bar{p} \vee \bar{q}) \wedge (p \vee q)$--from Example 5, using Method 2

Both translations are correct, but one or the other may be preferable in a given situation--for example, when constructing an electrical circuit. This will be made clear in the next section.

EXAMPLE 6:

INPUTS				OUTPUT
p	q	r	s	
0	0	1	1	0
all others				1

The sentence is:

"The output is 0 only when

p is 0	and	q is 0	and	r is 1	and	s is 1."
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\bar{p}$	$\wedge$	$\bar{q}$	$\wedge$	r	$\wedge$	s

The switching function is

$$\overline{\bar{p} \wedge \bar{q} \wedge r \wedge s}$$

which simplifies in this way:

$$\begin{aligned} \overline{\bar{p} \wedge \bar{q} \wedge r \wedge s} &= \overline{\bar{p} \wedge \bar{q}} \vee \overline{r \wedge s} \\ &= \bar{\bar{p}} \vee \bar{\bar{q}} \vee \bar{r} \vee \bar{s} \\ &= p \vee q \vee \bar{r} \vee \bar{s} \end{aligned}$$

PROBLEM SET 7:

In Problems 1-8 you will be using Method 1 to find switching functions. You can refer to page 32 for a summary of this method.

1. In the table at right the output is 1 when p is __?__ and q is __?__.

2. What is the switching function?

INPUTS p	q	OUTPUT
1	1	0
1	0	0
0	1	1
0	0	0

3. What is the switching function for the table at right?

INPUTS p	q	OUTPUT
1	1	0
1	0	0
0	1	0
0	0	1

4. In the table below, the output is 1 when p is __?__ and q is __?__ and r is __?__.

5. What is the switching function?

INPUTS p	q	r	OUTPUT
1	1	1	0
1	0	1	1
0	1	1	0
0	0	1	0
1	1	0	0
1	0	0	0
0	1	0	0
0	0	0	0

6. What is the switching function for the following table?

INPUTS p	q	r	OUTPUT
1	1	1	0
1	0	1	0
0	1	1	1
0	0	1	0
1	1	0	1
1	0	0	0
0	1	0	0
0	0	0	0

7. What is the switching function for the following table?

INPUTS p	q	r	OUTPUT
1	0	1	1
0	1	0	1
all others			0

8. What is the switching function for the following table?

INPUTS p	q	r	s	OUTPUT
1	0	0	1	1
all others				0

In the next problems you will need to use Method 2 for finding switching functions. See page 33 for a summary of the method.

9. In the table at right the output is 0 when p is __?__ and q is __?__.

10. What is the switching function? [Remember that you must take the negation of a statement and then simplify it by De Morgan's law.]

INPUTS		OUTPUT
p	q	
1	1	1
1	0	1
0	1	0
0	0	1

11. What is the switching function for the table at right?

INPUTS		OUTPUT
p	q	
1	1	1
1	0	1
0	1	1
0	0	0

12. In the table at right the output is 0 when p is __?__ and q is __?__ and r is __?__.

13. What is the switching function?

INPUTS			OUTPUT
p	q	r	
1	1	1	1
1	0	1	1
0	1	1	1
0	0	1	0
1	1	0	1
1	0	0	1
0	1	0	1
0	0	0	1

14. What is the switching function for the table at right?

INPUTS			OUTPUT
p	q	r	
1	1	1	1
1	0	1	0
0	1	1	1
0	0	1	1
1	1	0	1
1	0	0	0
0	1	0	1
0	0	0	1

In the following problems you will need to use both methods to find switching functions.

15. Use Method 1 to find the switching function for the table at right.

16. Use Method 2 to find the switching function for the table at right.

INPUTS		OUTPUT
p	q	
1	1	1
1	0	0
0	1	0
0	0	1

17. Use Method 1 to find the switching function for the table at right.

18. Use Method 2 to find the switching function for the table at right.

INPUTS p	INPUTS q	OUTPUT
1	1	0
1	0	1
0	1	1
0	0	1

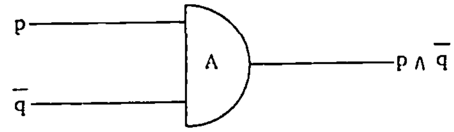
SECTION 8: SWITCHING FUNCTIONS TO CIRCUITS

8-1 <u>Introductory Examples</u>

This section is the payoff for the whole logic sequence. Here you will find out how to design a logic circuit having a given switching function. This and the preceding section, taken together, will provide you with the skills needed to design a circuit that corresponds to a given truth table--a circuit that will allow you to solve biomedical problems in your Science course.

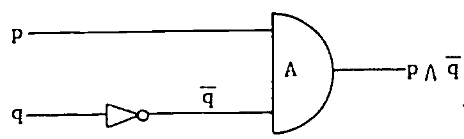
Suppose we wish to design a circuit for the switching function $p \wedge \bar{q}$. We can get $p \wedge \bar{q}$ as the <u>output</u> of an AND gate with <u>inputs</u> p and $\bar{q}$.

STEP 1



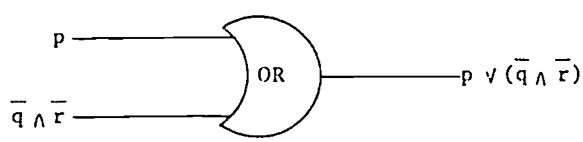
Working back to the basic inputs, p and q, we can get $\bar{q}$ as the <u>output</u> of an INVERT gate with <u>input</u> q.

STEP 2



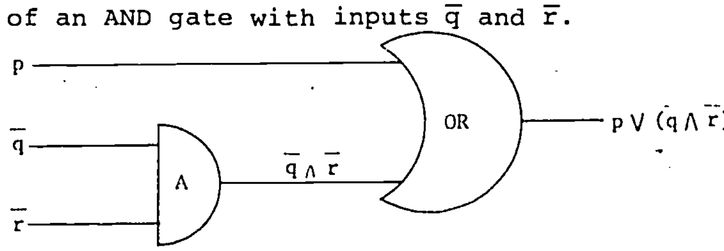
Let's try a more complicated case. Suppose we want a circuit with switching function $p \vee (\bar{q} \wedge \bar{r})$. We can get this as the <u>output</u> of an OR gate with <u>inputs</u> p and $\bar{q} \wedge \bar{r}$.

STEP 1



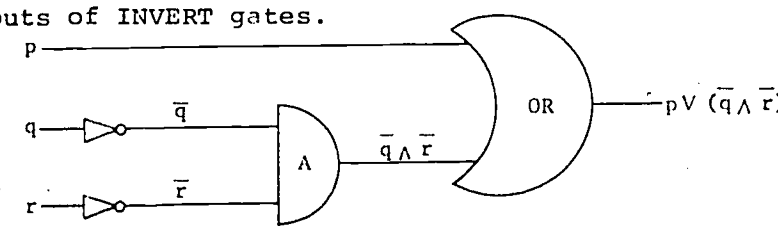
We then get $\bar{q} \wedge \bar{r}$ as the output of an AND gate with inputs $\bar{q}$ and $\bar{r}$.

STEP 2



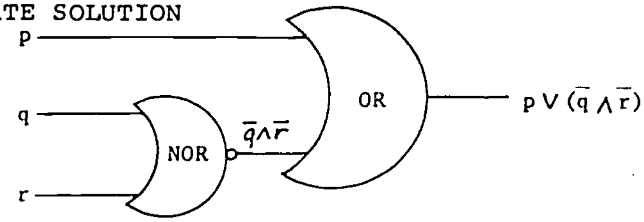
And we get $\bar{q}$ and $\bar{r}$ as the outputs of INVERT gates.

STEP 3



But notice that we could also get $\bar{q} \wedge \bar{r}$ as the output of a NOR gate with <u>inputs</u> q and r. The output of the NOR gate is $\overline{q \vee r}$. By De Morgan's law this is the same as $\bar{q} \wedge \bar{r}$.

ALTERNATE SOLUTION



Both the solution in Step 3 and the alternate solution are equally correct. Which one we use will depend on practical circumstances. For instance, we would use the first solution if we didn't have a NOR gate, or if NOR gates cost ten times as much as other gates. We would use the alternate solution if we wanted to use as few gates as possible. The important thing to notice about these two solutions is that the use of the NOR gate gives us a simpler circuit, with fewer gates.

8-2 Summary of the Technique

The technique used in the examples can be applied to any switching function. We can summarize the process this way.

1. Construct the circuit in a series of steps, starting with the output end.

   a. Ask yourself, "What inputs, applied to which gate, will give me this output?"

   b. The circuit is finished when you have worked down to the basic input letters, p, q, r, etc.

2. This method will always work, using only the AND, OR and INVERT gates.

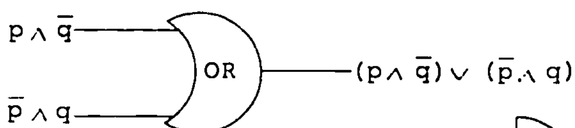
3. If NAND and NOR gates are available, there is more than one way to construct the circuit. A good way to design a circuit with the fewest possible number of gates is to remember the switching functions for NAND and NOR gates and use those gates whenever possible.

8-3 Some More Examples

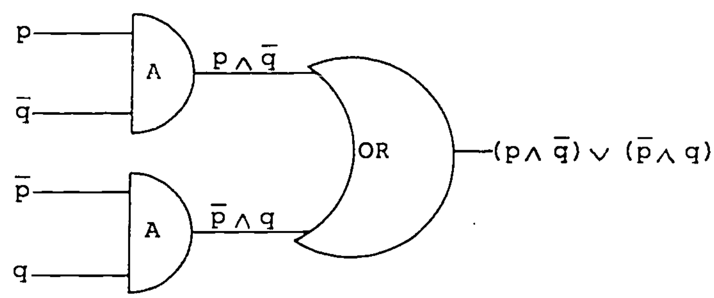
EXAMPLE 1: Construct a circuit with switching function $(p \wedge \bar{q}) \vee (\bar{p} \wedge q)$.

SOLUTION:

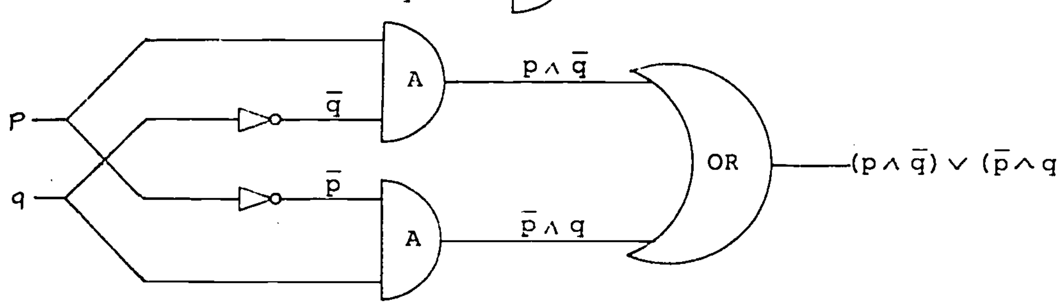
STEP 1



STEP 2



STEP 3

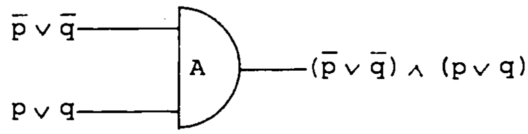


Notice that in order to provide two p inputs, one merely brings two wires off from the p input wire. The same is true for the q inputs. Although a p input wire and q input wire cross in the diagram, we assume that they are not electrically connected.

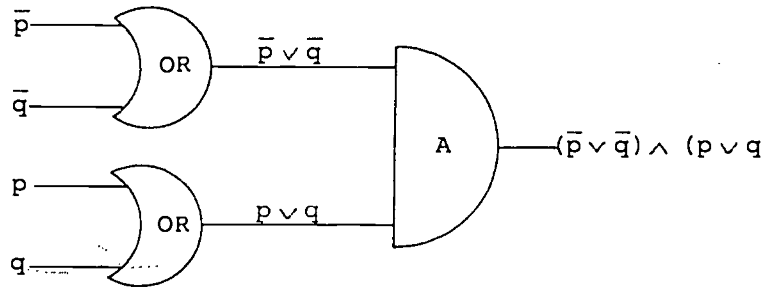
<u>EXAMPLE 2</u>: Construct a circuit with switching function $(\bar{p} \vee \bar{q}) \wedge (p \vee q)$.

<u>SOLUTION</u>:

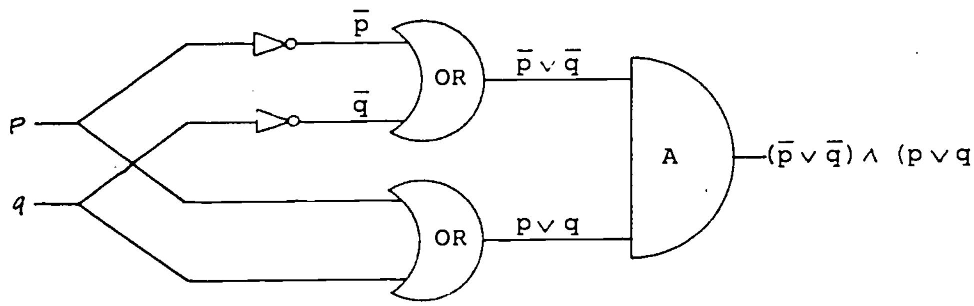
STEP 1



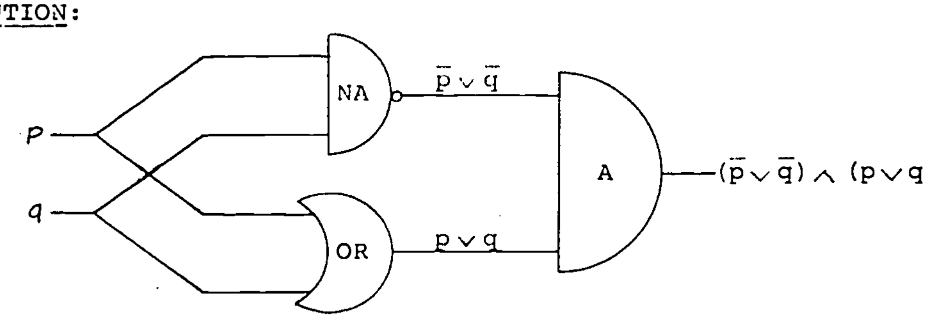
STEP 2



STEP 3



<u>ALTERNATE SOLUTION</u>:



The switching functions for these last two examples are

$$(p \wedge \bar{q}) \vee (\bar{p} \wedge q)$$

$$(\bar{p} \vee \bar{q}) \wedge (p \vee q)$$

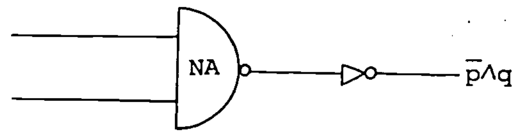
In Examples 2 and 5 of Section 7 we saw that these are equivalent translations of <u>exclusive or</u>; they both have the same truth table.

The three circuits we have just designed are quite different. But they all "do the same thing"--they all represent <u>exclusive or</u>. The choice of which circuit is best will depend on cost and availability of the various kinds of gates.

In Science class you will often be designing circuits with a NAND gate and INVERT gate at the output end. Such circuits are convenient in diagnosis problems.

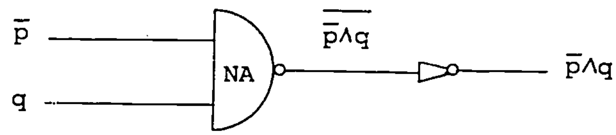
EXAMPLE:

Complete the circuit at right.

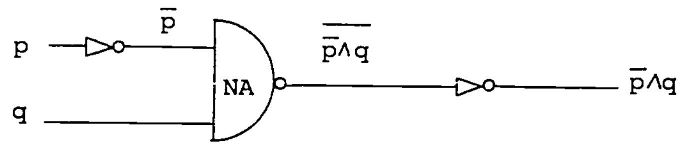


SOLUTION:

Switching functions on the opposite sides of an INVERT gate are always the negative of each other. Therefore the output of the NAND gate must be $\overline{\overline{p}\wedge q}$. Since this is just "not ($\overline{p}$ and q)," the inputs of the NAND gate must be $\overline{p}$ and q.



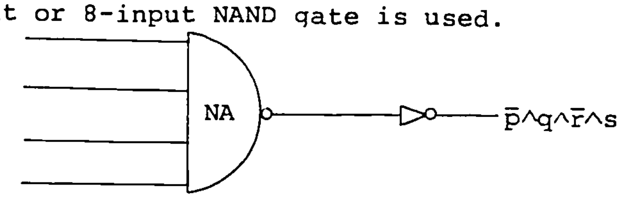
The last step involves inserting an INVERT gate in the p input wire.



The same approach works when a 4-input or 8-input NAND gate is used.

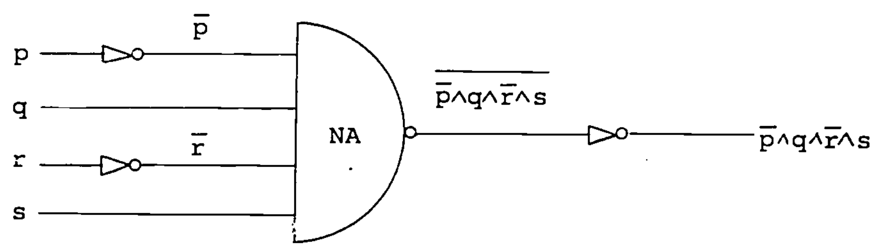
EXAMPLE:

Complete the circuit at right.



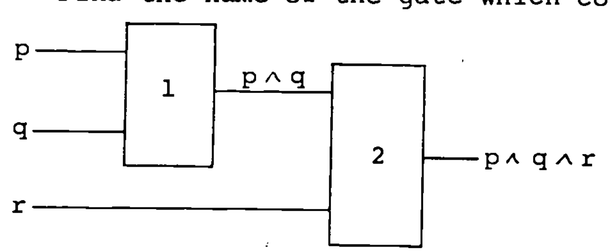
SOLUTION:

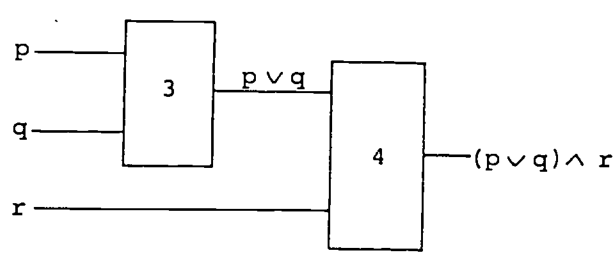
We work from right to left just as in the last example. The result is

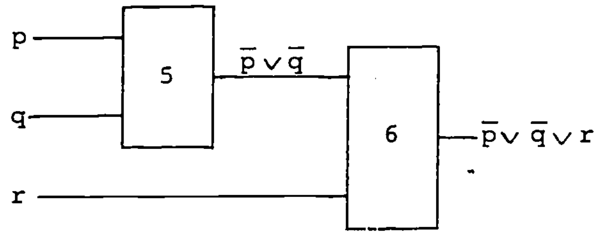


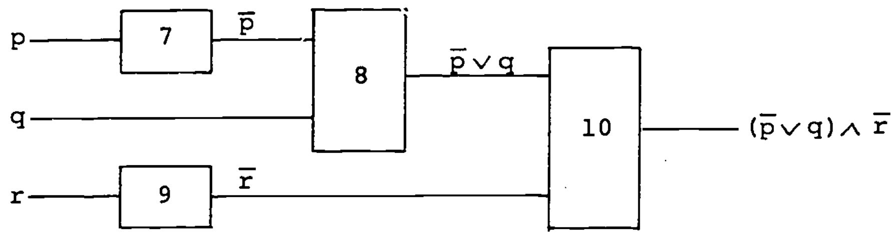
PROBLEM SET 8:

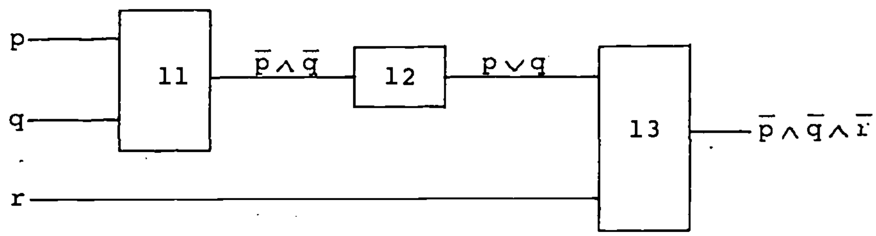
In the following circuits each numbered square represents one of the gates AND, NAND, OR, NOR, INVERT. Find the name of the gate which corresponds to each number.









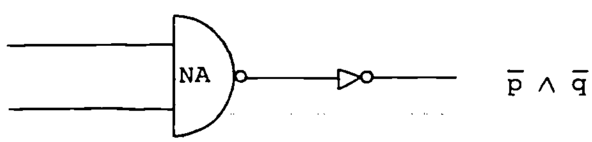


Draw circuit diagrams for the following switching functions.

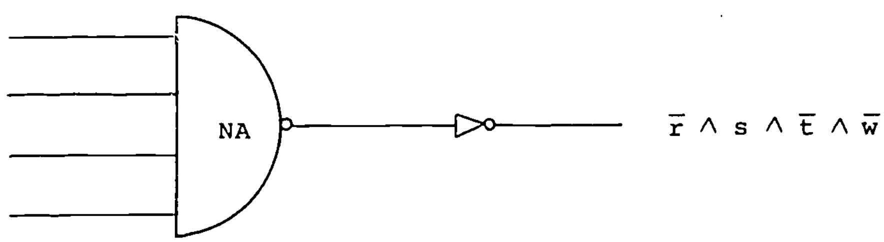
14. $\overline{p} \vee q$ 15. $p \vee (q \wedge r)$ 16. $\overline{p} \vee \overline{q} \vee \overline{r}$ 17. $(p \wedge r) \vee (\overline{p} \wedge s)$ 18. $p \wedge \overline{q} \wedge \overline{r} \wedge \overline{s}$

Complete the following circuits.

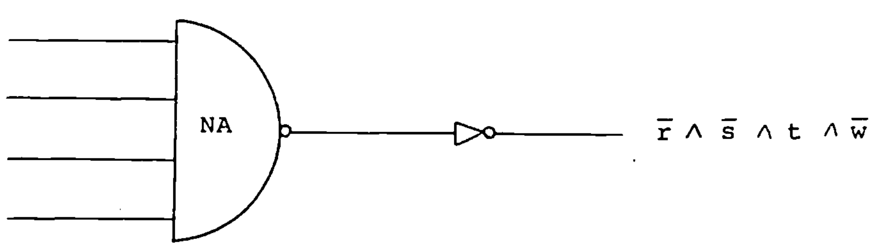
19.



20.



21.



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REVIEW PROBLEM SET 9:

For Problems 1 through 3, if the proposition is a statement, write "S." Otherwise write "NS." In addition, indicate by 1 or 0 the truth or falsity of those propositions which are statemnets.

1. In the United States, polio is no longer an important health problem.

2. In 1973 no one in the United States died.

3. Neither of the following is true.
   a. The moon is made of green cheese.
   b. The moon is earth's only satellite.

Suppose that the three statements below are given.

p: the patient has a high blood-sugar level.

q: the patient has a normal EEG.

r: the patient has normal blood pressure.

Put each of the following statements into symbolic language.

4. The patient has abnormal blood pressure.

5. The patient has a high blood-sugar level and doesn't have a normal EEG.

6. The following are both false.
   a. The patient has an abnormal EEG.
   b. The patient has a high blood-sugar level.

7. The patient has a normal EEG or the patient has a high blood-sugar level.

8. Are any of Statements 4 through 7 equivalent?

9. Complete the following truth table.

p	q	$\overline{q}$	$p\wedge\overline{q}$	$\overline{p\wedge\overline{q}}$
1	1			
1	0			
0	1			
0	0			

10. Complete the following truth table.

p	q	$\overline{p}$	$\overline{p}\vee q$
1	1		
1	0		
0	1		
0	0		

11. a. Are $\overline{p\wedge\overline{q}}$ and $\overline{p}\vee q$ equivalent statements? (Refer to Problems 9 and 10.)

    b. Justify your answer to Part a without the use of truth tables.

Simplify the following statements. Show your work.

12. $\overline{\overline{p}\vee\overline{q}}$

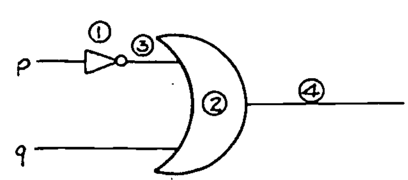
13. $\overline{p\wedge\overline{q}}$

14. $\overline{p\wedge\overline{q}\wedge r}$

*15. $\overline{(\overline{p}\vee q)\wedge\overline{(\overline{p}\wedge q)}}$

16. a. What are the names of gates ① and ② in the circuit to the right?

    b. What are the switching functions ③ and ④?



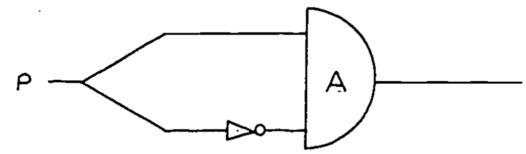
c. Complete the truth table at right for the circuit in Part a.

d. What is the output of the circuit in Part a when the input vector [p, q] is [1, 0]?

p	q	3	output 4
1	1		
1	0		
0	1		
0	0		

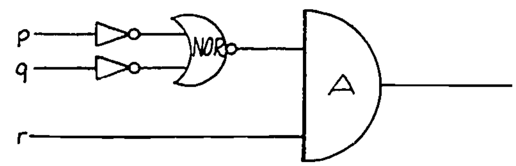
17. a. What is the output of the circuit at right when the input is [p] = 1?

b. For what input vector(s) [p] is the output 1?



18. a. Write down all input vectors [p, q, r] that will give an output of 1 in the circuit to the right.

b. Design a simpler circuit with the same output as the circuit in Part a.

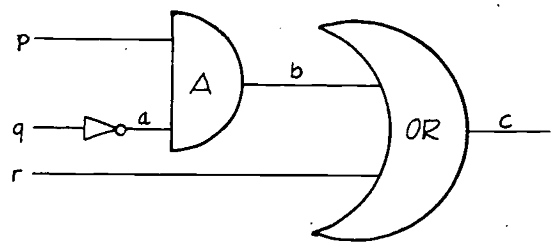


19. In the circuit on the right,

a. Switching function a is ___?___.

b. Switching function b is ___?___.

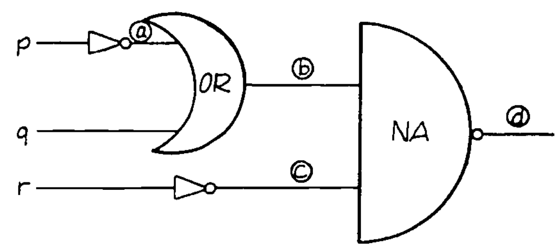
c. Switching function c is ___?___.



20. What is the output vector of the circuit in Problem 19 when the input vector is [1, 1, 0]?

21. Provide the lettered switching functions in the circuit at right.

22. Are the circuits in Problems 19 and 21 equivalent? Justify your answer.



What are the switching functions for the following truth tables? Simplify when necessary.

23.

INPUTS		OUTPUT
p	q	
1	1	0
1	0	0
0	1	1
0	0	0

24.

INPUTS		OUTPUT
p	q	
1	1	1
1	0	1
0	1	0
0	0	1

25. a. Draw a circuit diagram for the switching function found in Problem 23.

b. Draw a circuit diagram for the switching function found in Problem 24.

c. Do both the circuits that you drew in a and b above use the same number of gates?

d. Is that number 2?

e. If not, draw circuits with the same outputs but using only 2 gates in each circuit.

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What are the switching functions for the following truth tables? Simplify when necessary.

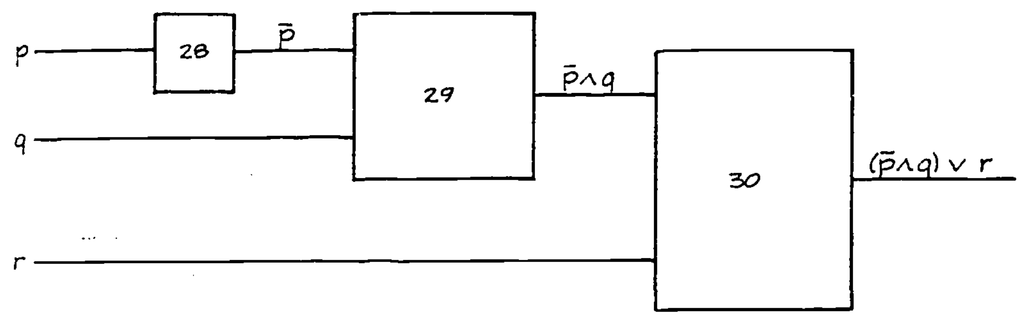
26.

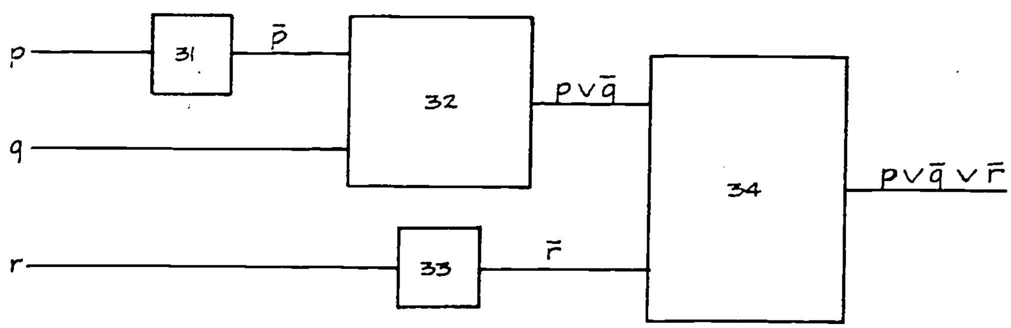
INPUTS			OUTPUT
p	q	r	
1	1	1	1
0	0	0	1
all others			0

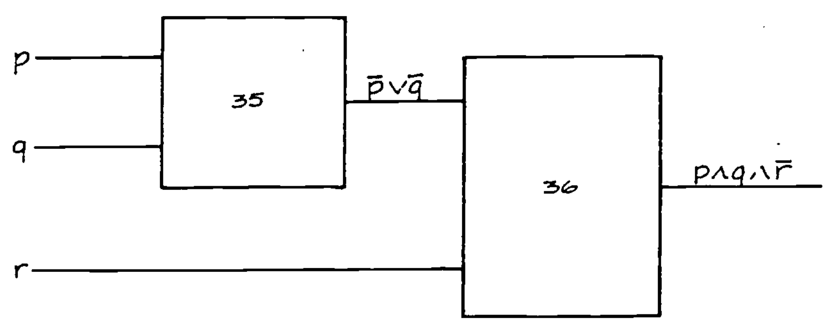
27.

INPUTS			OUTPUT
p	q	r	
1	1	1	0
0	0	0	0
all others			1

In the following circuits, each numbered square represents one of the gates AND, NAND, OR, NOR, INVERT. Find the name of the gate corresponding to each number.







Draw circuit diagrams for the following switching functions.

37. $p\vee\bar{q}$ 38. $\bar{p}\wedge q$ 39. $\bar{p}\wedge\bar{q}$ 40. $(p\wedge\bar{q})\vee(\bar{p}\wedge q)\vee(\bar{p}\wedge\bar{q})$

41. a. Did you use two or more gates to draw the circuit in Problem 39?

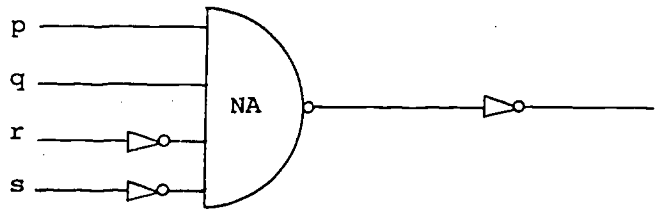
b. If you did, draw another circuit having the same output, but using only one gate.

42. a. How many gates did you use to draw the circuit of Problem 40?

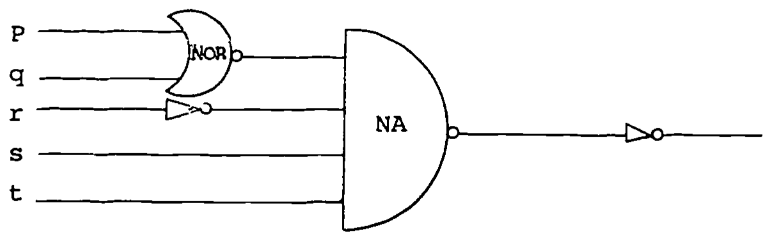
*b. If you used more than one gate, redraw the circuit using only one gate.

What input vectors will give an output of 1 in each circuit below?

43.

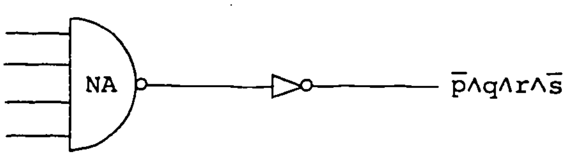


44.

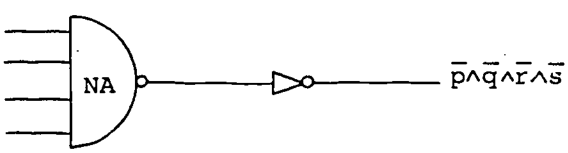


Complete each of the following circuit diagrams.

45.



46.



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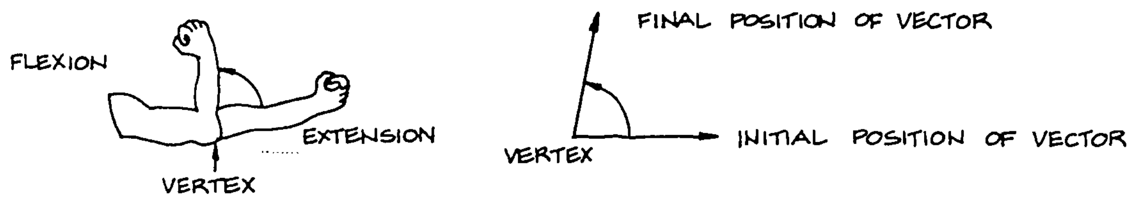
SECTION 10: ANGLES AND ANGLE MEASURE

10-1 Angles

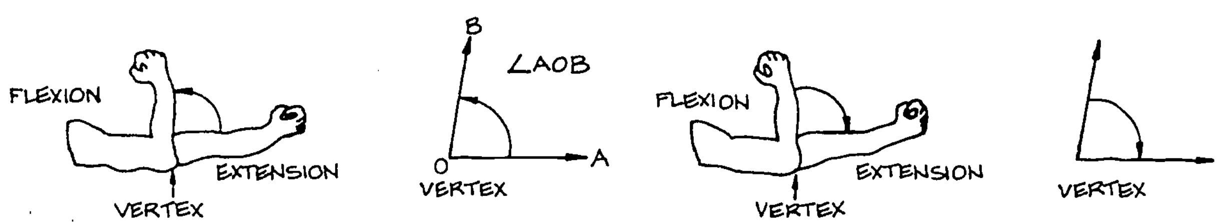
Trigonometry is the study of the relationships between the sides and angles of triangles. But the applications of trigonometry go far beyond triangles. For example, trigonometry can be applied to the forces exerted by muscles, to the bending of light by a lens and to the shape of a sound wave.

The study of trigonometry begins with angles. The human body may be used to generate many angles. Consider the case of Elmo flexing his muscles as shown below.

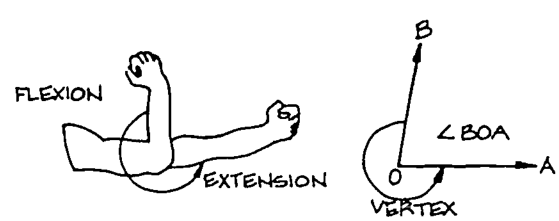


As Elmo goes from extension to flexion, he generates a positive angle (from our point of view) because the rotation is counterclockwise. The point about which the rotation occurs is called the vertex. In Elmo's case, the vertex is his elbow. On the right above is a diagram of how an angle is represented by mathematicians. An angle is considered to be the geometric figure generated when a vector is rotated about its tail.

Below, we have Elmo generating both positive and negative angles as he flexes his muscles for a group of admirers.



When we designate an angle, "$\angle AOB$," we mean that the angle is generated by rotating the head of a vector from A to B in a counterclockwise direction as is shown in the left-hand figure above. It is important when designating an angle to write the letters in the proper order because $\angle AOB \neq \angle BOA$. The figure to the right shows Elmo attempting to generate $\angle BOA$ when somebody gave him an angle with the letters in an awkward order for the abilities of his arm.



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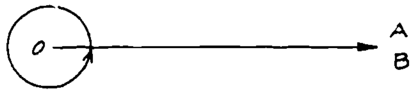
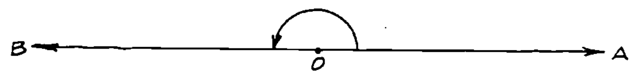
The rotational capabilities of the various parts of the body are of biomedical significance. As part of the activities connected with this lesson, you will have the opportunity to measure some of the angles and compare your angles to the statistical averages obtained by measuring large numbers of people.

10-2 Measuring Angles

Angles are commonly measured in units called degrees. A vector that is rotated until it returns to its original position generates an angle of 360 degrees (left-hand figure below).



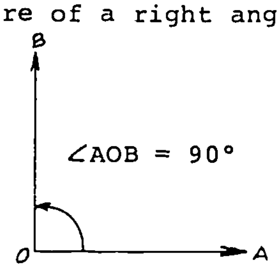
∠AOB = 360°

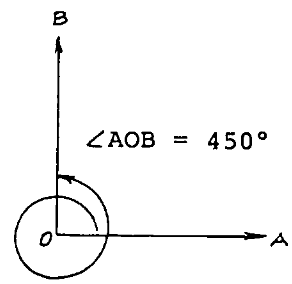


∠AOB = 180°

If a vector is rotated until it points in the opposite direction, it generates an angle whose measure is half of 360 degrees, or 180 degrees (right-hand figure above).

The measure of a right angle is 90 degrees.

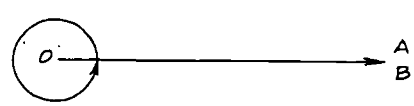




We may also speak of angles greater than 360 degrees. The angle resulting from the rotation of a vector through 360 degrees and then an additional 90 degrees measures 450 degrees.

In the discussion above we have done something that is not strictly formal mathematically. In Section 10-1 we defined an angle to be a geometric figure. In this section we have equated a geometric figure with its measure in degrees. This is similar to saying, "Triangle A equals 1 m^2," when the strictly formal thing to say is, "The area of triangle A is 1 m^2." However, to be strictly formal in this situation requires extra verbiage and notation which in itself clutters up the equations and explanations unnecessarily we feel. What you should remember is that there is an important distinction between a geometric figure and its measure even though we won't be calling it to your attention through either the language we use or any special notation.

Angles may also be measured in units of revolutions. One revolution is equal to 360 degrees (see figure to right); 720 degrees = 2 revolutions; 180 degrees = $\frac{1}{2}$ revolution. You are probably familiar with units of revolutions per minute (RPM) as measures of the rate of revolution of automobile engines and phonograph records. For example, typical speeds for records are 45 RPM and 33 RPM. A clinical centrifuge spins at speeds from 1000 to 2000 RPM.



∠AOB = 1 revolution

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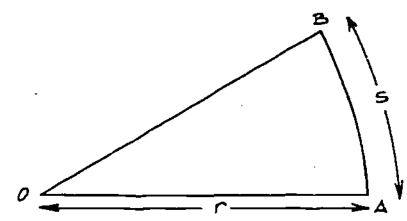
A measure of angles related to revolutions is cycles. Mathematically, 1 cycle = 1 revolution. The idea of a cycle refers to a repetitive pattern, e.g., 1 year is the time required for one revolution of the earth about the sun and also the time required for one cycle of the seasons. You may be interested to learn that the word "cycle" comes from the Greek word for "circle." Therefore, the word "bicycle" translates as "two circles."

10-3 Radian Measure

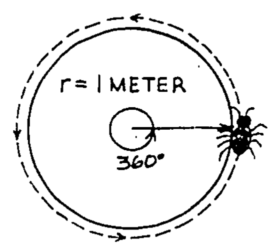
Another unit used to measure angles is seen less commonly than the degree but is more useful in many mathematical and scientific problems. This unit is the radian. First we will define a radian, then explain it.

We will define a radian in terms of a circular arc (an arc is a portion of a circle). The number of radians in an angle is the length of the circular arc that the angle cuts off (s in the figure to the right) divided by the radius of the circular arc (r). In the illustration to the right, the measure of $\angle AOB$ in radians is s divided by r.



$AOB = \frac{s}{r}$ radians

Suppose that a bug walks along a circle with a radius of 1 meter. The circumference of the circle is equal to 2π meters (because $c = \pi d = 2\pi r$). So when the bug has completed one trip around the circle (one revolution, or 360°), it has traveled a distance of 2π meters.



Distance Traveled

1 revolution = 1 cycle

$= 2\pi$ meters

≈ 6.28 meters

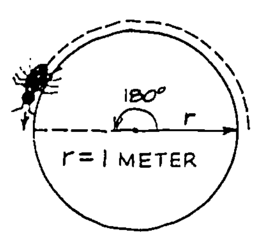
(because $c = \pi d = 2\pi r$)

The measure in radians of one revolution is the distance traveled by the bug divided by the radius of the circle.

$$\frac{2\pi \text{ meters}}{1 \text{ meter}} = 2\pi \text{ radians}$$

Therefore, One Revolution = 360° = 2π radians. Notice that there are units of meters in both the numerator and denominator on the left side of the equation. Ordinarily, these units would cancel, leaving no units on the right. However, radians appear as units on the right because, by definition, this is the way the units of radians are derived.

Now let the bug go on a journey halfway around the circle ($= \frac{1}{2}$ cycle). The arc of the bug's path corresponds to an angle of 180 degrees. The distance traveled by the bug is one-half the circumference, or



Distance Traveled

$\frac{1}{2}$ revolution $= \frac{2\pi \text{ meters}}{2}$

$= \pi$ meters

≈ 3.14 meters

$$\frac{1}{2} \cdot 2\pi \text{ meters} = \pi \text{ meters}$$

The radian measure of 180 degrees is again the distance the bug traveled divided by the radius.

$$\frac{\pi \text{ meters}}{1 \text{ meter}} = \pi \text{ radians}$$

180 degrees is equivalent to π radians.

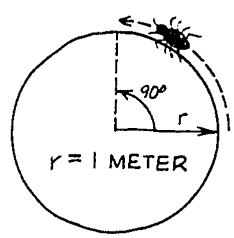
The equivalence of 360 degrees to 2π radians and of 180 degrees to π radians leads to a conversion factor for converting degrees to radians. The conversion factor is

$$\frac{\pi \text{ radians}}{180 \text{ degrees}}$$

Since π radians and 180 degrees are equal, multiplying by the conversion factor is the same as multiplying by one.

If the bug walks one-quarter of the way around the circle, its path subtends an angle of 90°. We may use our conversion factor to convert a measure of 90 degrees to a measure in radians.

$$90 \text{ degrees} \cdot \frac{\pi \text{ radians}}{180 \text{ degrees}} = \frac{\pi}{2} \text{ radians}$$



Distance Traveled

$$\frac{1}{4} \text{ cycle} = \frac{2\pi \text{ meters}}{4}$$
$$= \frac{\pi}{2} \text{ meters}$$
$$\approx 1.57 \text{ meters}$$

To show that the conversion factor gives the correct answer, we may also calculate the radian measure of 90 degrees as the length of the arc divided by the radius.

One-fourth of the circumference is

$$\frac{1}{4} \cdot 2\pi r = \frac{\pi r}{2}$$

and the radian measure is this distance divided by the radius, by definition.

$$\frac{\pi r}{2} \cdot r = \frac{\pi}{2} \text{ radians}$$

We may also convert a measure in radians to one in degrees. The conversion factor is

$$\frac{180 \text{ degrees}}{\pi \text{ radians}}$$

To demonstrate this conversion, let us convert a measure of $\frac{\pi}{3}$ radians into degrees.

$$\left(\frac{\pi}{3} \text{ radians}\right) \cdot \frac{180 \text{ degrees}}{\pi \text{ radians}} = 60 \text{ degrees}$$

If you wish to check this answer, let the bug travel one-sixth of the way around the circle ($\frac{1}{6}$ cycle or revolution).

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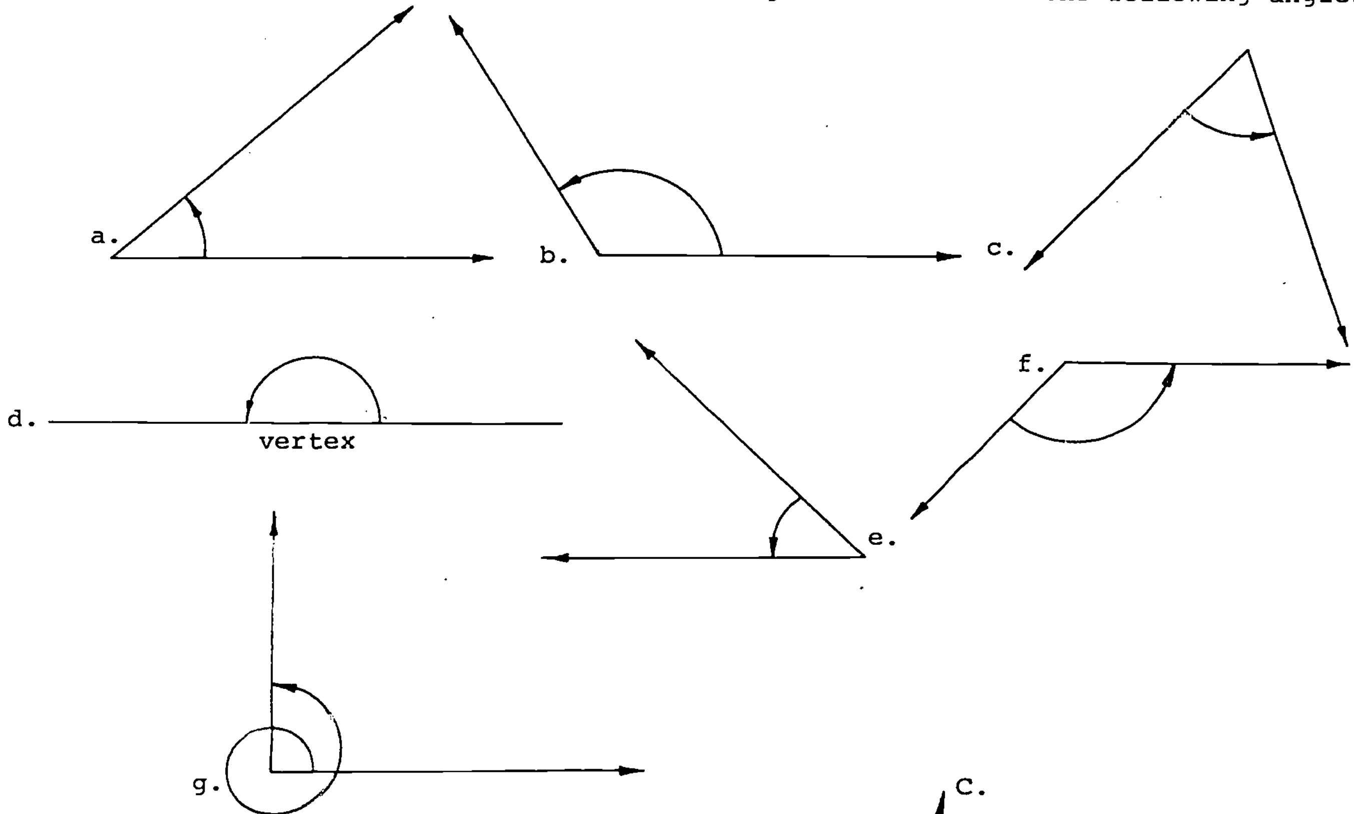
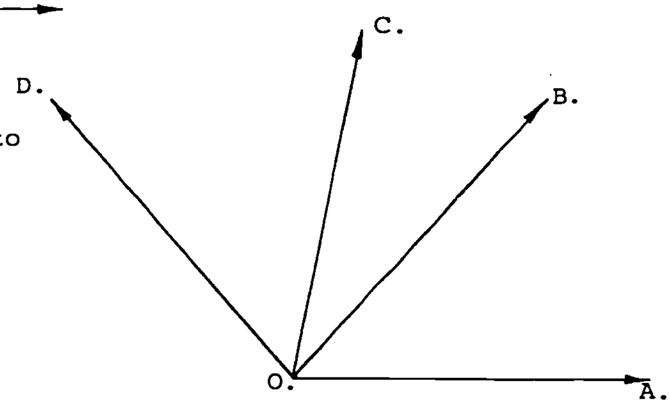
PROBLEM SET 10:

1. Using a protractor, find the number of degrees in each of the following angles.



2. Refer to the diagram at right to answer the following.



a. ∠ AOB = ?

b. ∠ BOC = ?

c. ∠ COD = ?

d. ∠ AOD = ?

e. ∠ BOA = ?

f. Does the sum of ∠AOB, ∠BOC and ∠COD = ∠AOD?

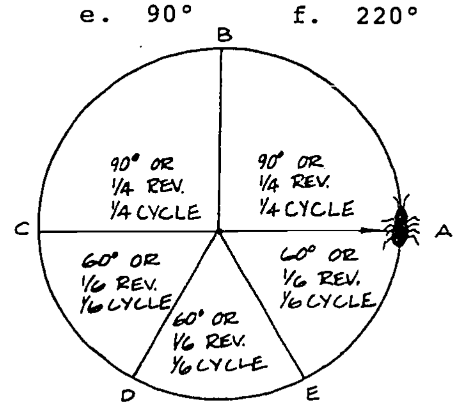
3. With your protractor, construct an angle equal to each of the following.

a. 30° b. 45° c. 120° d. 180° e. 90° f. 220°

4. Suppose the bug in the figure to the right is condemned to travel around the circle forever.

a. After the bug makes one complete cycle, how far has it traveled? Remember that the length around the edge of the circle is called the circumference, and circumference = $2\pi r$. Include units.

b. If the bug moves from A to B, how far has it traveled?



radius = 1 meter

c. If the bug moves from A to C, how far has it traveled?

d. The distance along the edge of the circle from A to D is __________.

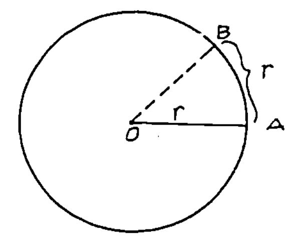
e. The distance along the edge of the circle from A to E is __________.

f. $\frac{\pi}{4}$ meters around the edge of the circle is equivalent to _____ degrees, or _____ revolutions.

g. $\frac{2\pi}{3}$ meters around the edge of the circle is equivalent to _____ degrees, or _____ revolutions.

h. $\frac{7\pi}{4}$ meters around the edge of the circle is equivalent to _____ degrees, or _____ revolutions.

5. In the circle at the right the radius is r. Suppose that the arc AB also has length r. The angle AOB that intersects the arc AB equal in length to the radius r is called a __________.

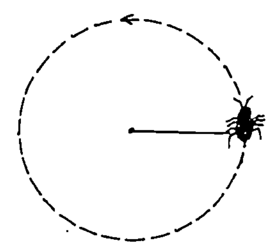


6. If the bug at the right makes one revolution,

a. How many degrees has it traveled around the circle?

b. How many radians has it traveled?

c. Thus ___?___ degrees equals ___?___ radians.



7. If 2π radians equals 360°, then

a. π radians = ______ b. $\frac{\pi}{2}$ radians = _____ c. $\frac{3\pi}{4}$ radians = _____

8. If 360° equals 2π radians, then

a. 90° equals _?_ radians. d. 220° equals _?_ radians.

b. 180° equals _?_ radians. e. 330° equals _?_ radians.

c. 30° equals _?_ radians.

9. Express each of the following both in degrees and in radians.

a. 2 revolutions c. $2\frac{1}{2}$ revolutions e. $\frac{7}{6}$ revolutions

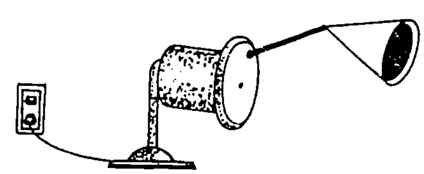
b. .75 cycle d. $\frac{2}{3}$ cycle f. 1.25 cycles

10. Through how many radians does the minute hand of a clock rotate in

a. 20 minutes b. 45 minutes c. 90 minutes d. 75 minutes

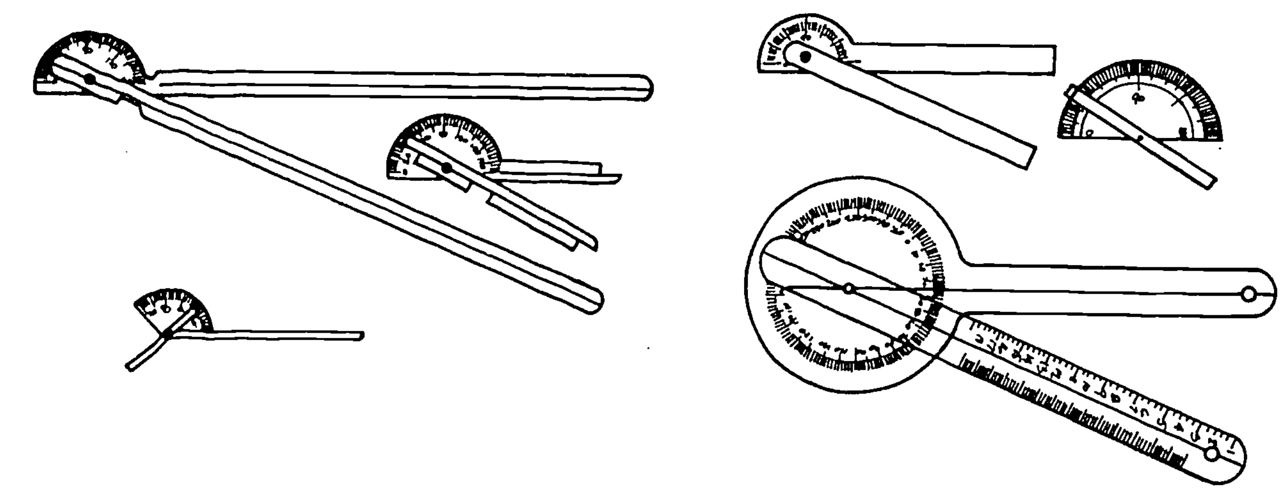
*11. A clinical centrifuge is running at 1200 revolutions per minute. Through what angle, in radians, does it rotate in 1 second?

*12. A small motor is connected to a speaker cone as shown to the right. As the motor spins, it pushes the speaker cone in and out, producing a sound. The motor spins at the rate of 60 revolutions per second. The speaker cone vibrates in and out at the rate of 60 cycles per second. What is the rate of revolution for the motor in radians per second?



ACTIVITY 10:

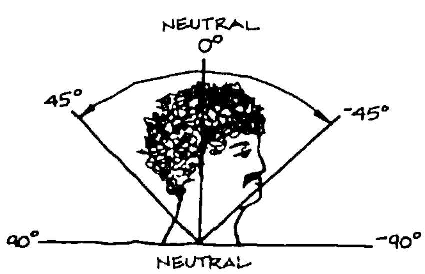
Many health practitioners such as orthopedic nurses, orthopedic surgeons, and physical therapists routinely measure the "range of motion" for body joints. The "range of motion" is the maximum angle a person is able to generate by moving a limb. The instruments used for these measurements are called "goniometers." A few of the available types are depicted below. They are essentially protractors with extension arms of some kind.



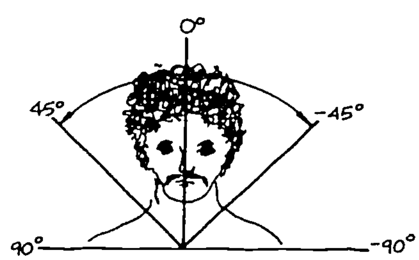
The instrument that you will use will likely be fashioned from meter sticks, and a protractor will be used to measure the angle.

The angles given in the diagrams are statistical means--averages, in other words--derived from measurements of large numbers of people. Since you and your classmates are young and flexible you will probably have little trouble generating the ranges of motion that are given. However, if a measured range of motion is 10 per cent* or more less than indicated, perhaps you should consult a physical therapist, orthopedic nurse or physician.

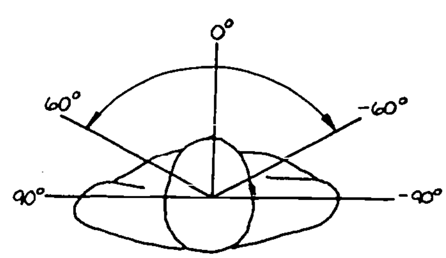
1. Measure the range of motion of the cervical spine. Record your measurements as ranges. For example, in the flexion and extension of the cervical spine illustrated below, suppose Elmo could nod his head forward 50° and backward 40°. This motion would be recorded as 40° to -50°.



FLEXION AND EXTENSION



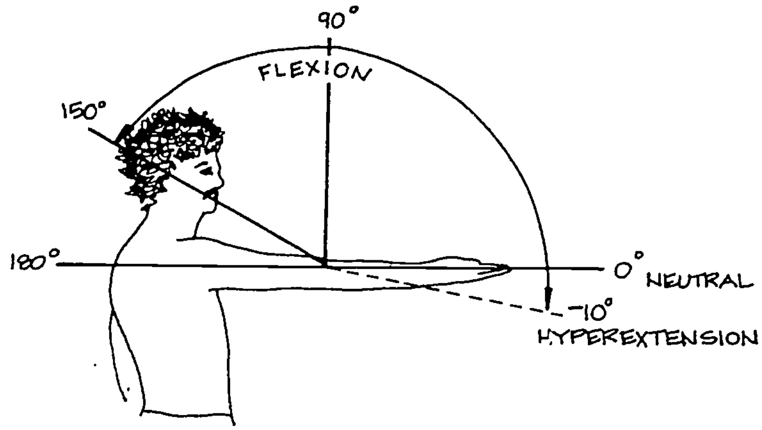
LATERAL BEND



ROTATION

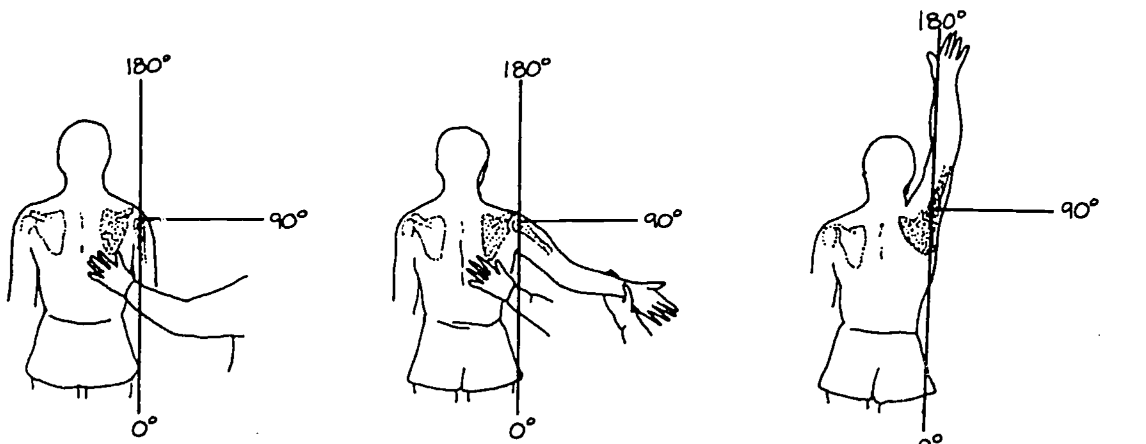
*Note: This is a guess.

2. Measure the range of motion of the elbow. Record your results for flexion and hyperextension separately.



FLEXION AND HYPEREXTENSION

3. Measure the ranges of motion for the shoulder as shown below.

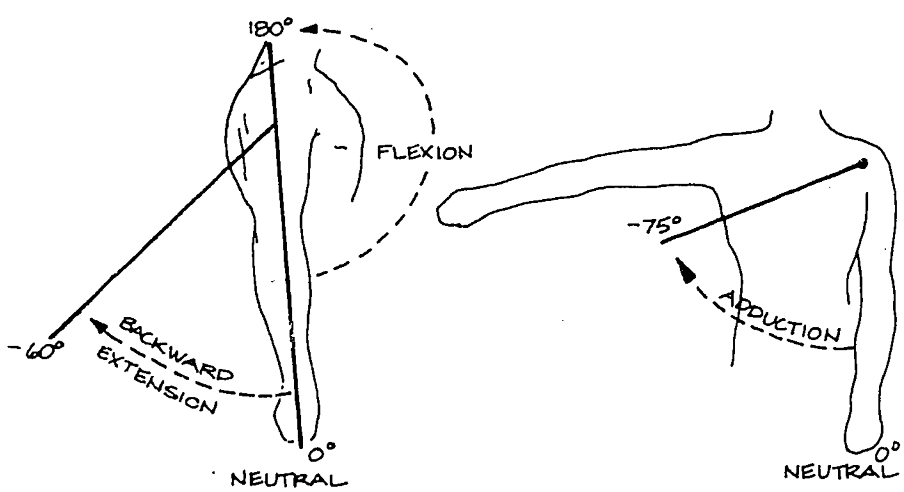


NEUTRAL

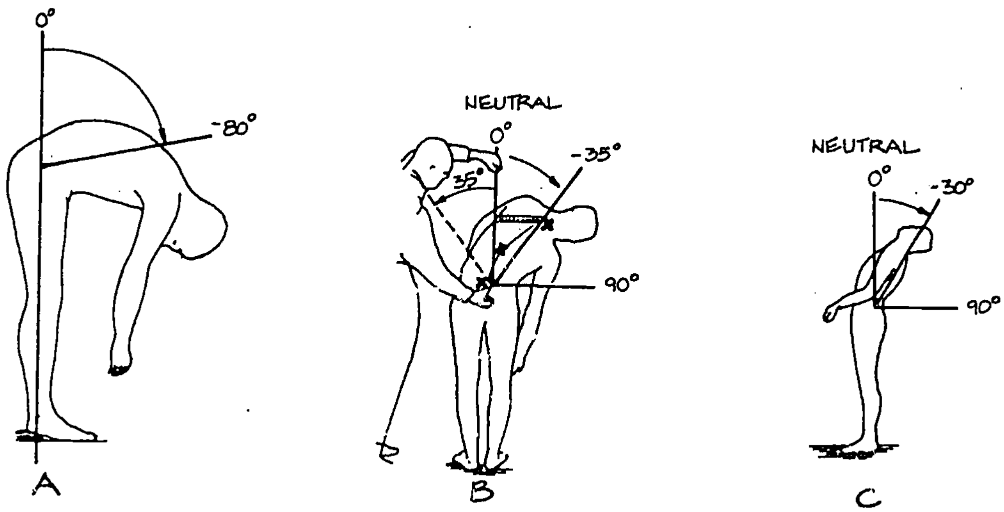
RANGE OF TRUE GLENOHUMERAL MOTION

"COMBINED" GLENOHUMERAL AND SCAPULOTHORACIC MOTION

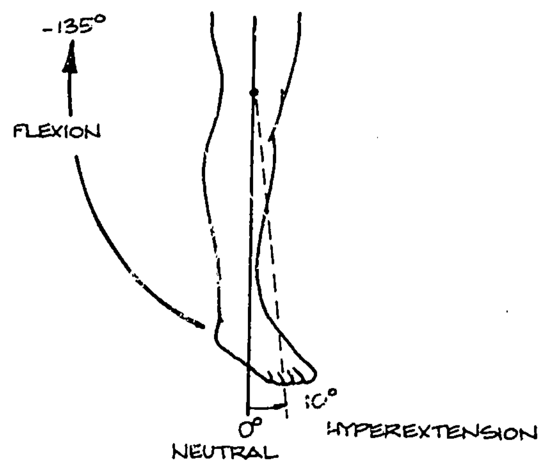
4. Measure the ranges of motion of the shoulder as shown below.



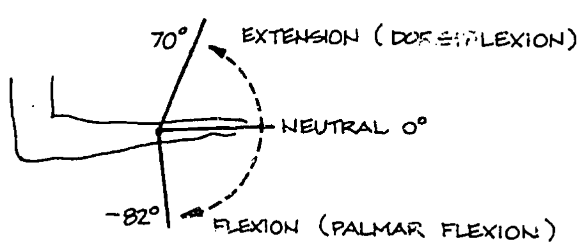
5. Measure the three different bending modes of the back which are illustrated below.



6. Measure the range of motion of the knee. Record flexion and hyperextension separately.



7. Measure the range of motion of the wrist.



ERIC

SECTION 11: SIMILAR FIGURES

11-1 Some Examples

The concept of similarity is a central one in the study of trigonometry. Furthermore, an understanding of the properties of similarity are required in the analysis of X-rays.

On the following two pages are sets of figures. In each set one of the figures is similar to the figure labeled "ORIGINAL." The rest are distorted in some fashion. Before you read Section 11-2 we ask you to do two things.

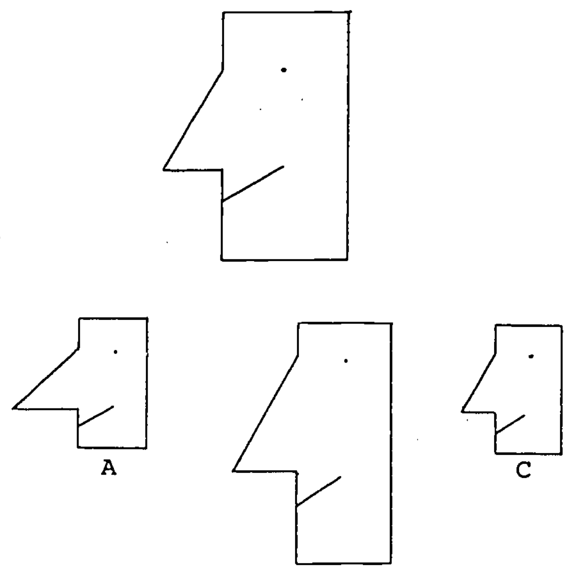
First, we ask you to examine each set and identify the figure which appears to be an undistorted reproduction of the original, that is, similar to the original in every way except size.

Second, we ask you to try to think of some mathematical rules which could be used to identify similar figures and eliminate distorted ones. Use a ruler and a protractor to measure the figures. In what way are the measurements of two similar figures related?

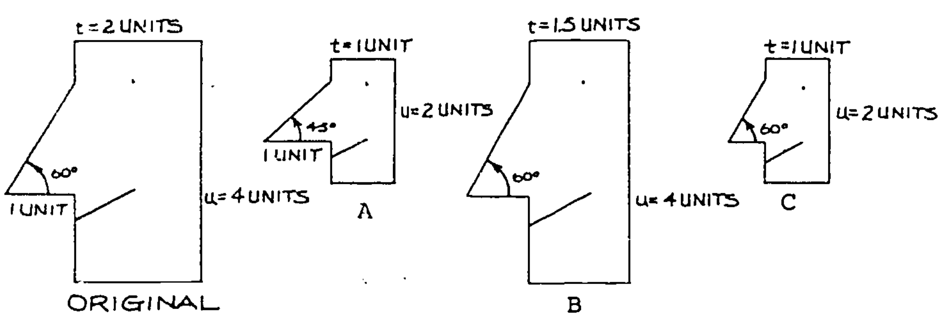
11-2 Similar Figures

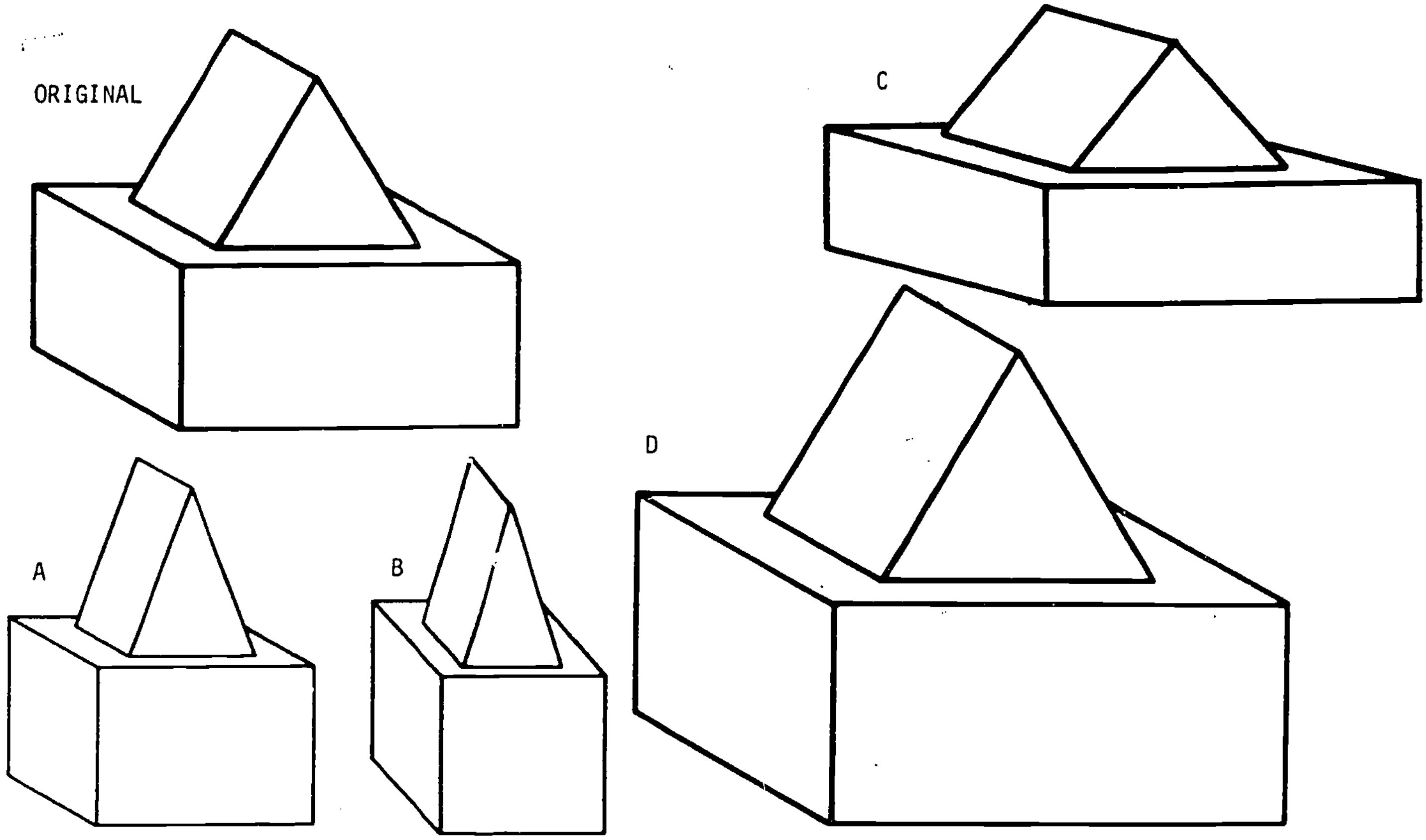
The first illustration to the right is a rather clumsy attempt at the profile of a human face. Below that are three more profiles, apparently by the same artist. Which of these three drawings represents the same person that the drawing above them does?

If your answer is C, you are correct, but how did you know? Can you state exactly why you think A and B are not the same person as the original?

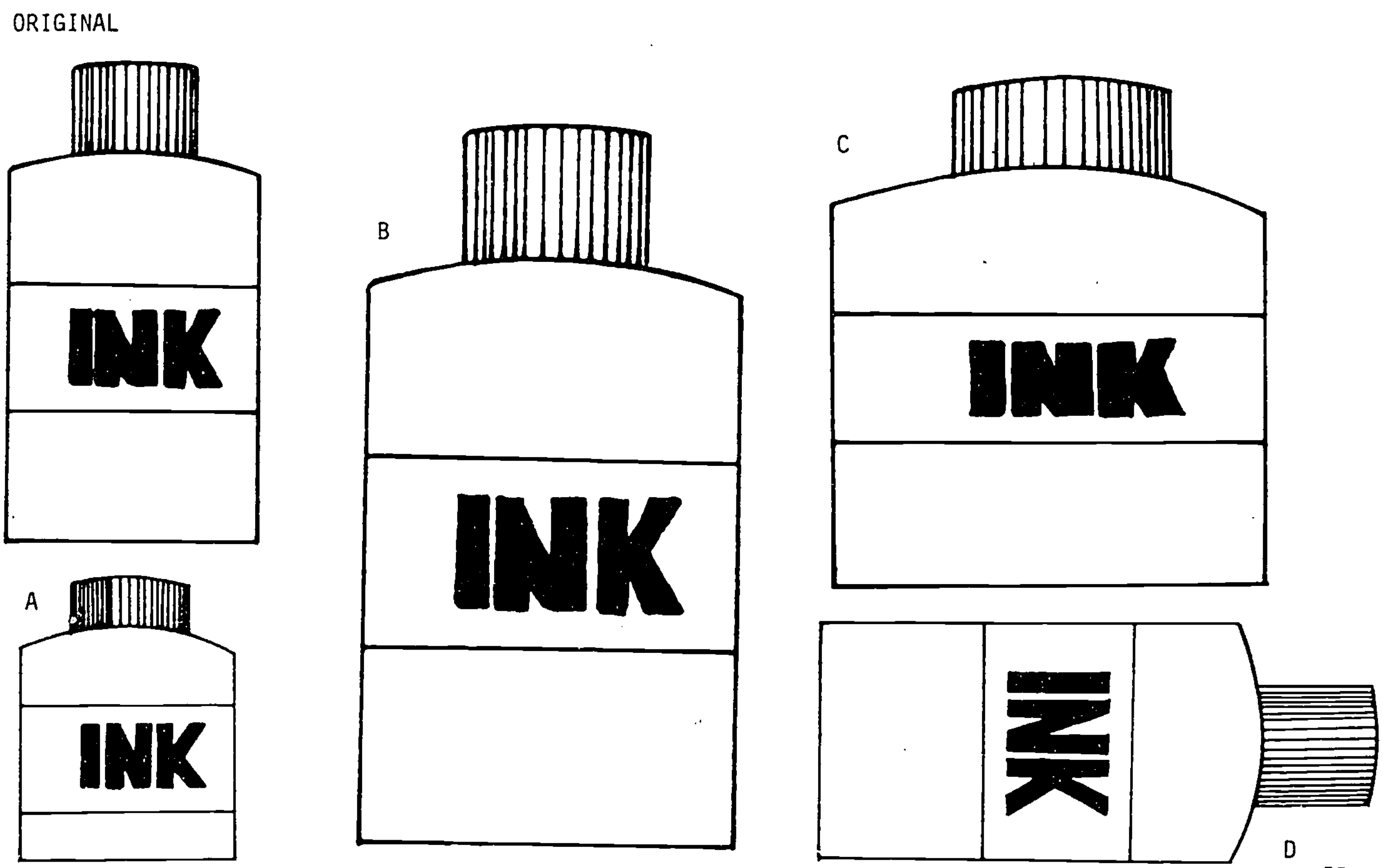


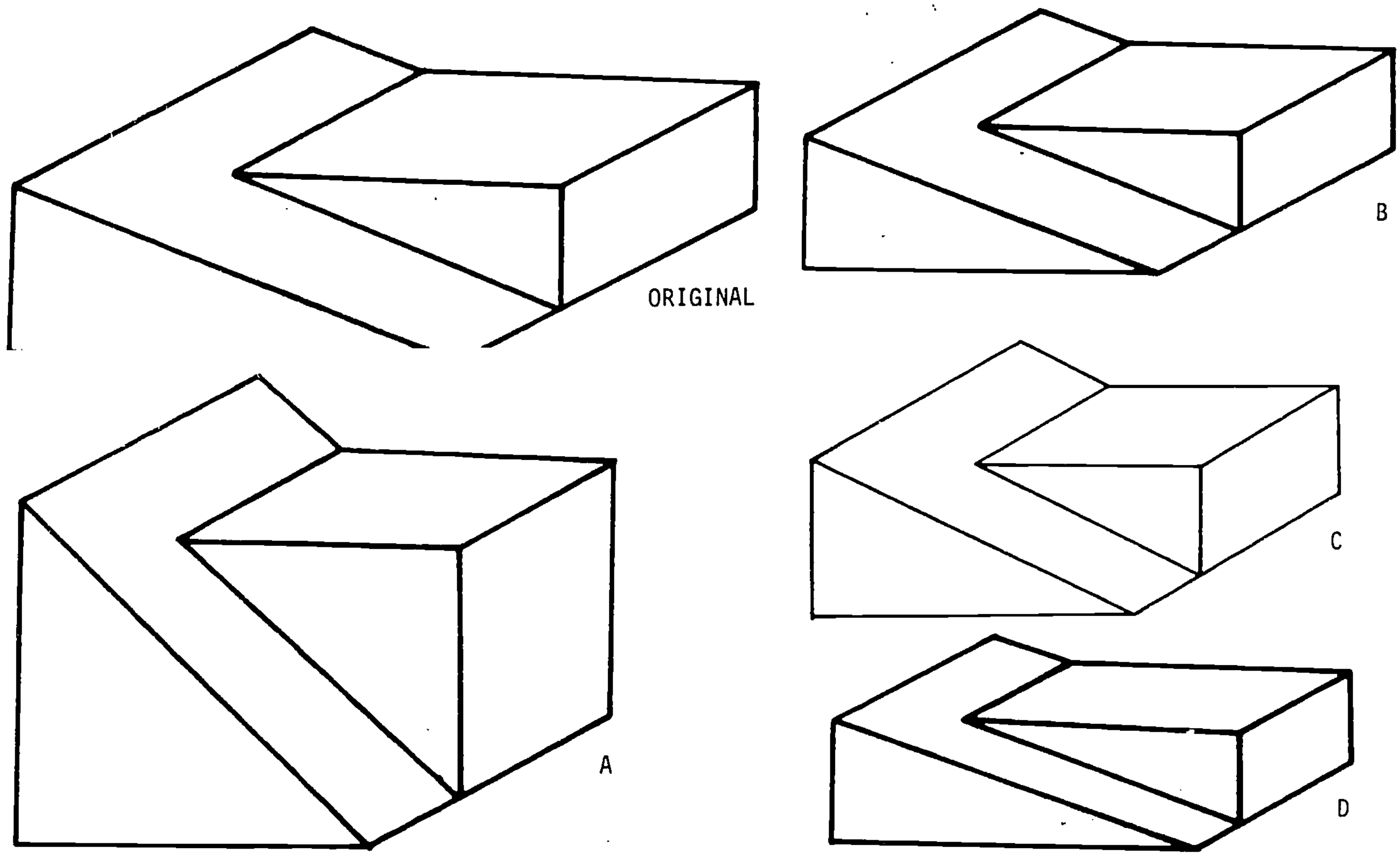
Let us label a few of the dimensions. The drawings are shown again below with the lengths of the heads from front to back (t), the heights of the heads (u) and the angles of the noses indicated.

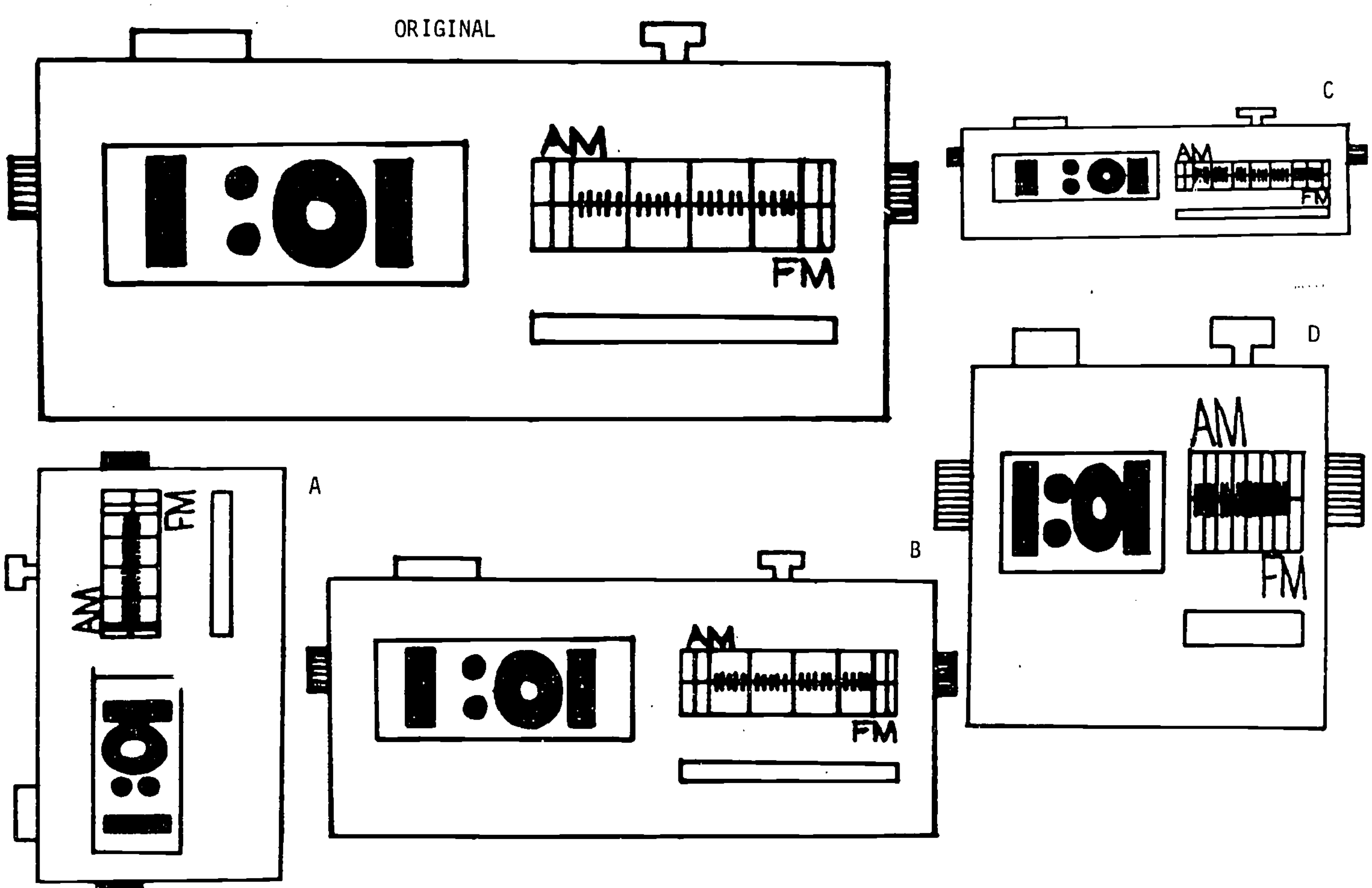




WORKSHEET 11




ORIGINAL
A
B
C
D


ORIGINAL
AM
FM
A
B
C
D

Consider A. Doesn't his nose look more pointed than the original? That it is more pointed may be confirmed by comparing the nose angles in the two figures. A's nose angle is 45° while the original's is 60°.

What about the ratios of the heights of the heads to their lengths from front to back? The ratio of u to t in the original is $\frac{4}{2} = 2$. The ratio in A is $\frac{2}{1} = 2$. In B the same ratio is $\frac{4}{1.5} = 2\frac{2}{3}$. And in C, height u divided by length t is $\frac{2}{1} = 2$.

The ratios of u to t in A and C are the same as in the original; the ratio in B is different, however.

A is unlike the original because the angle within the nose is not the same. B is unlike the original because the ratio of the height of the head to the length is different. Only C has the same nose angle and ratio of height to length as the original. C is smaller than the original, but it could be the same person drawn to a different "scale."

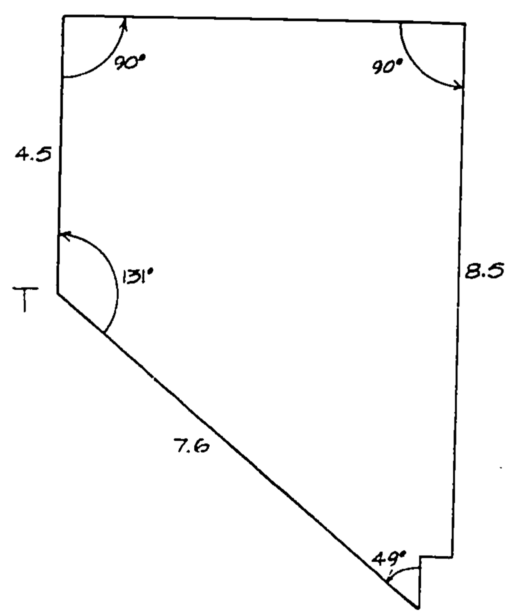
The idea of scale suggests another way to look at C, the similar figure. Notice that each length of C is half the corresponding length of the original. This feature of the relationship between the original and C could be used as a test for similarity. A is unlike the original because the ratio of widths is $\frac{2}{1}$ while the ratio of nose lengths is $\frac{1}{1}$, an unlike ratio. B is unlike the original because the ratio of widths is $\frac{2}{1.5}$ while the ratio of heights is $\frac{4}{4} = \frac{1}{1}$.

The original and C are said to be similar figures. Two similar figures may be accurate representations of the same object as viewed from one location. In justifying our choice of C as the figure that is similar to A we have used three properties which are true of similar figures.

One property is that all the angles of one figure are equal to the corresponding angles of the other figure.

The second is that the ratio of any two lengths in one figure is equal to the ratio of the corresponding lengths in the other figure.

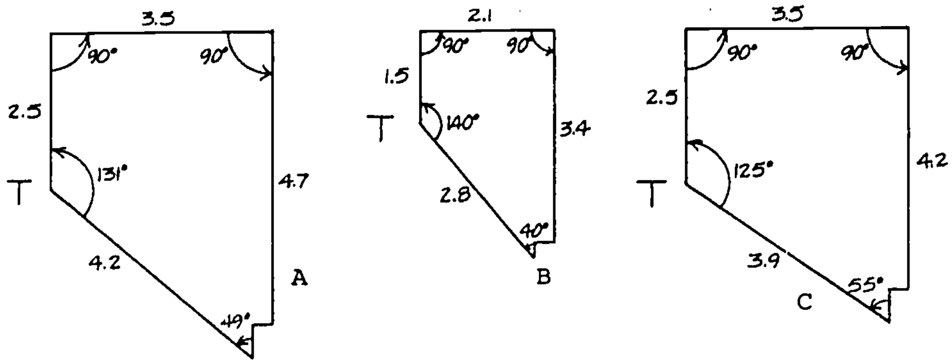
The third is that the ratio of the lengths of two corresponding sides is the same as the ratio of the lengths of any other two corresponding sides.



11-3 Similar to Nevada?

Let us apply our criteria to a few outlines of Western states. The first is a reasonably accurate representation of a map of Nevada.

The angles as well as the lengths of the sides are given (except for the lower-right corner which we will ignore). Below are three other figures that may or may not correctly represent Nevada.



We may determine which of these figures represents Nevada by comparing them to the first, which is known to be correct. If a figure is similar to the original, each of its angles is equal to the corresponding angles of the original.

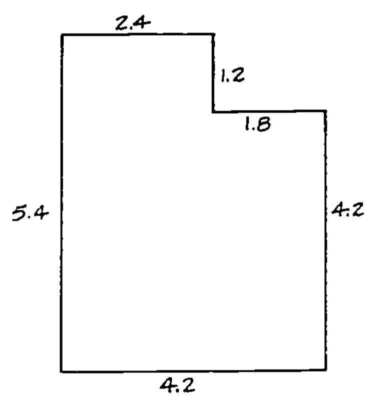
The angle on the western border (labeled T for Lake Tahoe) of A is 131°, the same as the original, but the corresponding angles of B and C are 140° and 125°. Therefore B and C are not similar to the original. All the angles in A, however, are equal to those of the original.

If A and the original are similar, their sides must be proportional. Let us see if they are. The ratio of the northern border of the original to the northern border of A is $\frac{6.3}{3.5} = 1.8$. The ratio of the western borders is $\frac{4.5}{2.5} = 1.8$; the ratio of the southwestern borders is $\frac{7.6}{4.2}$, or approximately 1.8; and the ratio of the eastern borders is $\frac{8.5}{4.7}$, which is also about 1.8.

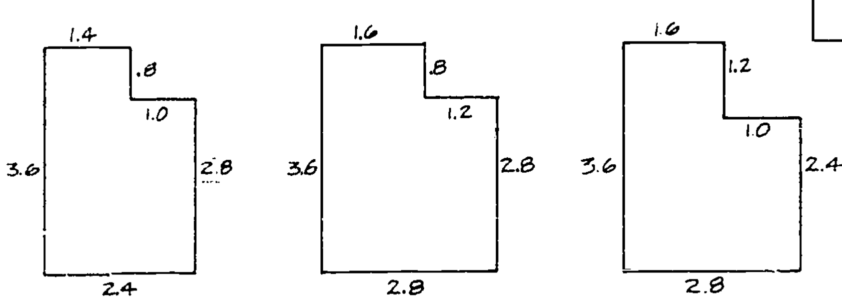
Consequently, the figures are similar, and A is a correct representation of the map of Nevada.

11-4 Similar to Utah?

Now let us take a look at Utah. An approximately correct representation of the Utah map is to the right.



The lengths of the various sides are given in arbitrary units. Which of the figures that are labeled A, B, and C (below) is also a correct map of Utah?



A B C

ERIC

Note that there is a 90° angle at each corner of all the figures. Let us then concentrate on the proportions of the sides.

The ratio of the western border of the original to the western borders of all three of the others is

$$\frac{5.4}{3.6} = \frac{3}{2}$$

However, the ratio of the southern border of the original map to the southern border of A is

$$\frac{4.2}{2.4} = \frac{7}{4}$$

$\frac{7}{4}$ is not equal to $\frac{3}{2}$; since their sides are not proportional, the correct figure and A are not similar.

The ratio of the southern border of the correct figure of Utah to the southern borders of B and C are in both cases

$$\frac{4.2}{2.8} = \frac{3}{2}$$

Therefore, B and C could both represent Utah, but let us check the dimensions in the upper right corner.

The upper eastern border of the original is

$$\frac{1.2}{.8} = \frac{3}{2}$$

times as long as the corresponding border of B, but is equal to the corresponding border of C. Therefore, C is not similar to the correct figure; the lengths of its western border and upper eastern border are not proportional to those of the original.

Every side of B that we have checked is proportional to the original. If you check the other sides, you will find them also proportional. Therefore B is similar to the original and correctly represents Utah.

If you would like another exercise with maps, determine whether Colorado and Wyoming are similar.

SECTION 12: SIMILAR TRIANGLES

12.1 Theorems for Similar Triangles

In the previous section we directed our attention to such diverse figures as a map of Utah and an absurd profile of a man's head. We mentioned several properties of similar figures, including equal angles and equal ratios for corresponding lengths. In this section we will restrict our attention to triangles, and we will treat similarity in a more formal manner.

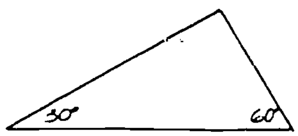
Let us begin by defining similarity in a formal way by using some of the criteria of Section 11. Two figures are similar if all corresponding angles are equal and all corresponding sides are of proportional length.

Does this definition mean that to know whether two triangles are similar we must know the lengths of all three sides and the measures of all three angles of each? No, because according to three theorems that we will state, similarity can be determined with less information than the dimensions of every side and every angle. We will not prove the theorems, but we will illustrate the use of each.

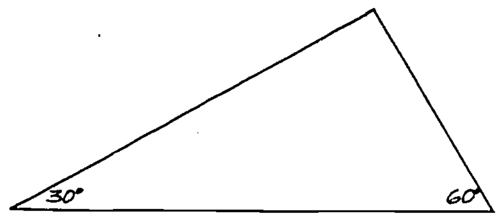
THEOREM 1: If two angles of one triangle are equal to the corresponding angles of another triangle, the triangles are similar. We will indicate this theorem with the symbol "AA" to remind you that this theorem concerns two angles in each triangle.

EXAMPLE:

Are the two triangles below similar?



A



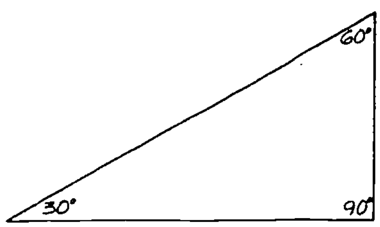
B

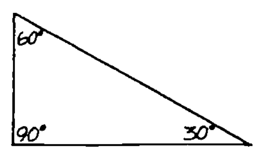
SOLUTION:

Two angles of triangle A are equal to two angles of triangle B. A is therefore similar to B.

EXAMPLE:

Are the two triangles below similar?





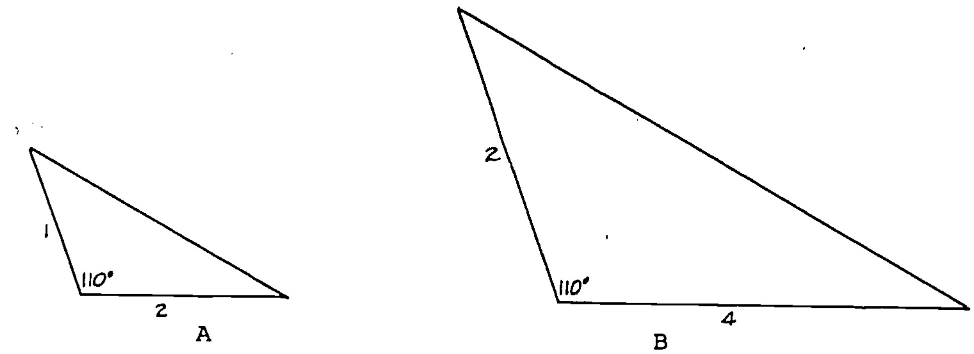
SOLUTION:

Even though these triangles have reversed or mirror image-like orientations, they are similar because of "AA."

THEOREM 2: If two sides of one triangle are proportional to two sides of another triangle, and if the angles between those sides are equal, the triangles are similar. This theorem is designated "SAS" for "Side-Angle-Side" to indicate that this theorem concerns two sides and the angle between.

EXAMPLE:

Are the two triangles below similar?



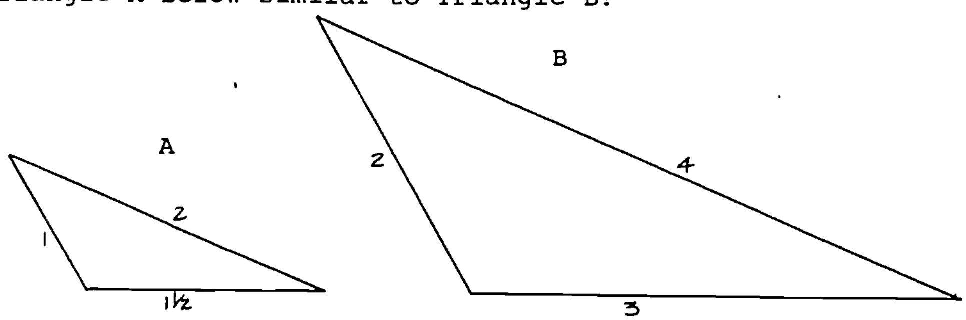
SOLUTION:

The two numbered sides of triangle B are twice the lengths of the corresponding sides of triangle A. The angle between these sides is 110° in both triangles. Therefore, according to Theorem 2, the two triangles are similar.

THEOREM 3: If the three sides of one triangle are proportional to the corresponding three sides of another triangle, the triangles are similar. You will see this theorem designates "SSS" for "Side-Side-Side," because it concerns the three sides of each triangle.

EXAMPLE:

Why is triangle A below similar to Triangle B?



SOLUTION:

Each side of triangle B is twice as long as the corresponding side of triangle A. Therefore these two triangles are similar.

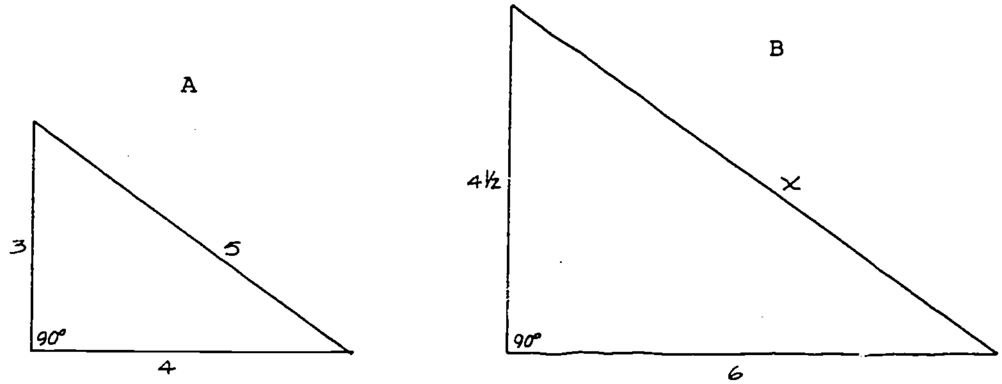
If you need to be convinced that any of the three theorems is true, try finding an exception. For example, try to draw a triangle with each of its sides proportional to the sides of another triangle but with an angle not equal to the corresponding angle of the other triangle.

12-2 Using Similarity to Find Unknown Parts

If two triangles are similar, the angles of one are equal to the corresponding angles of the other, and the corresponding sides are proportional. If two triangles are known to be similar, unknown parts of one may be determined from known parts of the other.

EXAMPLE:

We know the following two triangles are similar because of Theorem 2 (SAS). Find x.



SOLUTION:

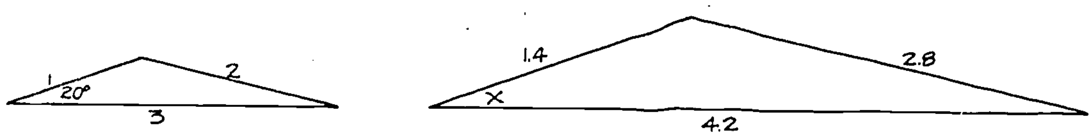
Two sides of triangle B are $1\frac{1}{2}$ times the lengths of the corresponding sides of triangle A, and the angle between these sides is 90° in both triangles.

We know that the third side of A is 5, but the third side of B is unknown. We do know, however, that since the triangles are similar, the unknown side of B, which we call x, is in the same ratio to the corresponding side of A as any other two corresponding sides. The length of the side corresponding to x is 5. Thus $\frac{x}{5} = \frac{6}{4}$, where $\frac{6}{4}$ is the ratio of two corresponding sides in the two triangles. We may solve this expression for x.

$$x = \frac{6}{4} \cdot 5$$
$$= 7\frac{1}{2}$$

EXAMPLE:

The triangles below are similar (SSS). Find angle x.

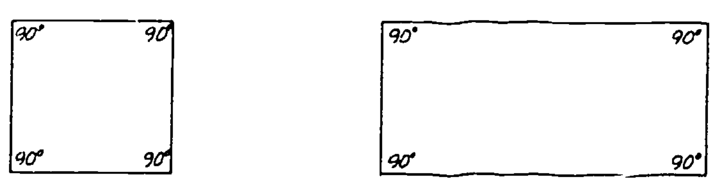


SOLUTION:

Since the triangles are similar, all corresponding angles are equal. Therefore, angle x = 20°.

12-3 May the Theorems Be Used for Other Figures?

The three theorems are valid only for triangles. For instance, the angles of a rectangle are equal to the corresponding angles of a square, but the two are not similar figures, because their corresponding sides are not proportional.



ERIC

And the two figures below, a square and a rhombus, have equal sides. Yet they are not similar, because the angles of one are not equal to the corresponding angles of the other.

The three similarity theorems are listed again below, since you may wish to refer to them while doing the problem set.

<u>Theorem 1</u> (AA): If two angles of one triangle are equal to the corresponding angles of another triangle, the triangles are similar.

<u>Theorem 2</u> (SAS): If two sides of one triangle are proportional to two sides of another triangle, and if the angles between those sides are equal, the triangles are similar.

<u>Theorem 3</u> (SSS): If the three sides of one triangle are proportional to the corresponding three sides of another triangle, the triangles are similar.

PROBLEM SET 12:

Certain words have been omitted in the statements numbered 1 through 5 below. Copy the sentences and include the appropriate words.

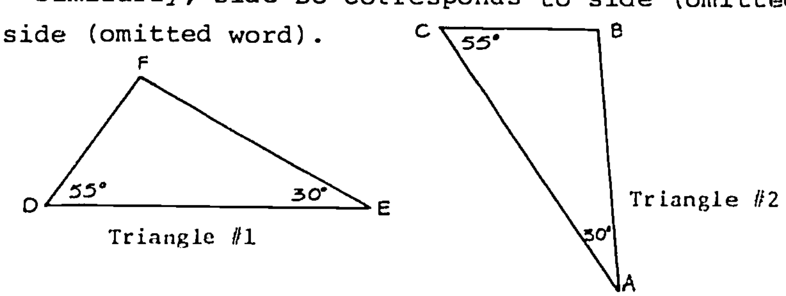
1. Two similar triangles have equal corresponding (omitted word) and proportional corresponding (omitted word).

2. If two (omitted word) of one triangle are equal to two corresponding (omitted word) of another triangle, the triangles are similar.

3. If two pairs of corresponding (omitted word) are proportional and the (omitted word) between them are equal, then the two triangles are (omitted word).

4. If three pairs of corresponding (omitted word) are proportional, then the triangles are (omitted word).

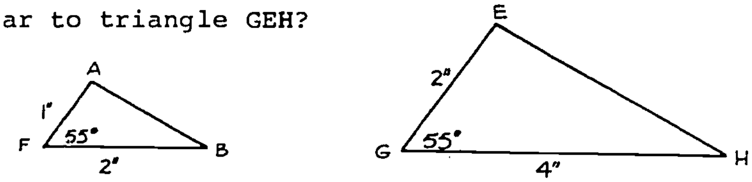
5. In the diagram below, side AB in Triangle #2 corresponds to side (omitted word) in Triangle #1. Similarly, side BC corresponds to side (omitted word) and side AC corresponds to side (omitted word).



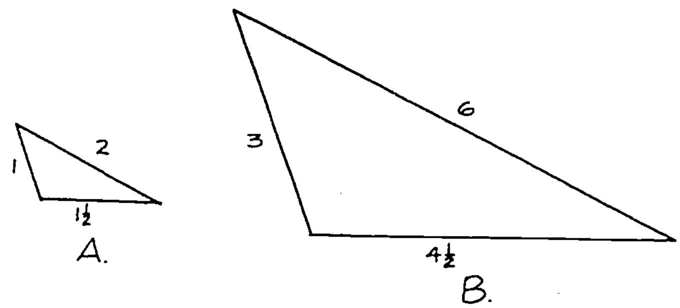
For Problems 6 through 8 state the appropriate theorem. (If you wish, you may use the proper abbreviation: AA, SAS or SSS.)

6. Refer to the two triangles of Problem 5. Why is triangle ABC similar to triangle EFD?

7. Why is triangle FAB similar to triangle GEH?



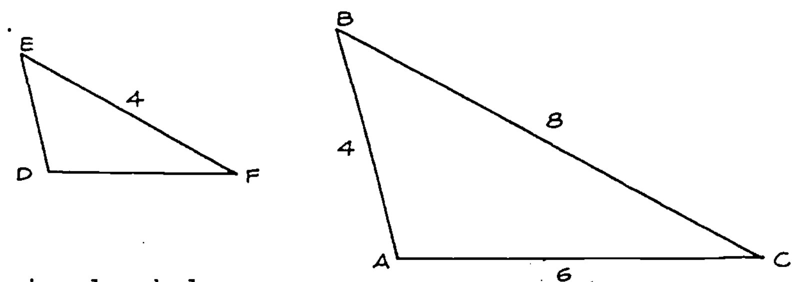
8. Why is triangle A similar to triangle B?



9. Given that the triangles below are similar,

a. DE = ?

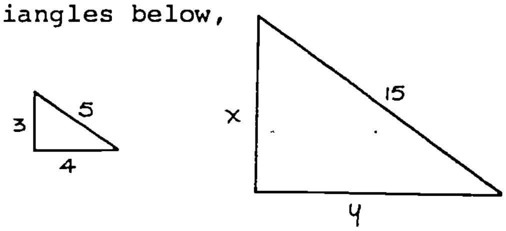
b. DF = ?



10. Given the similar triangles below,

a. x = ?

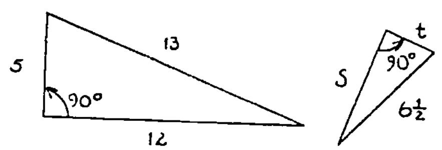
b. y = ?



11. Given the similar triangles below,

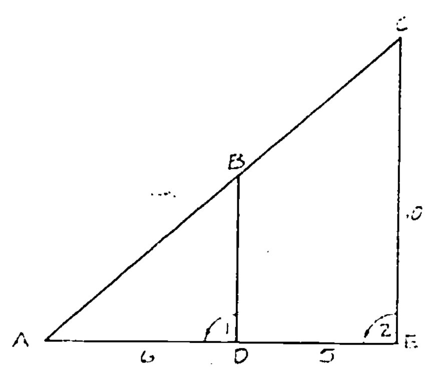
a. s = ?

b. t = ?



12. In the diagram at the right, if ∠1 and ∠2 are both right angles, why is triangle ADB similar to triangle AEC? (State theorem or abbreviation.)

[Hint: ∠A is the same angle in both triangles.]

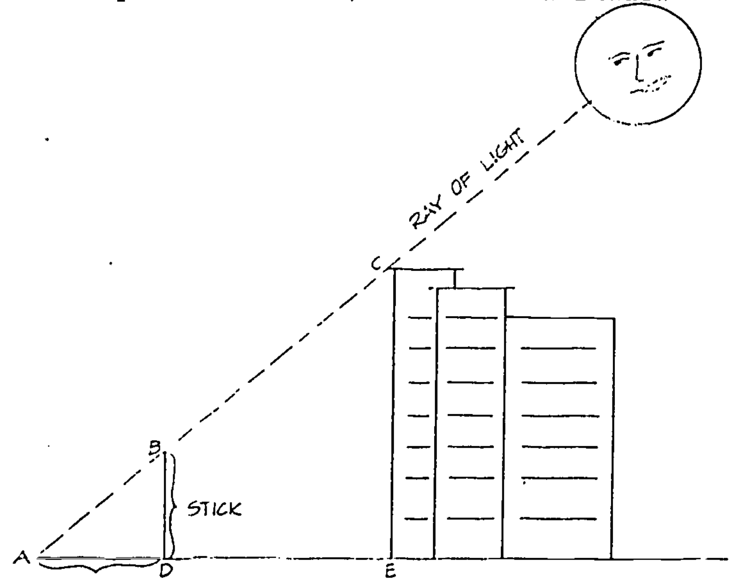


13. In the diagram for Problem 12, given that

a. triangle ADB is similar to triangle AEC, then $\frac{BD}{AD} = \frac{?}{AE}$.

b. If CE = 10, AD = 6, DE = 5, then BD = ?

14. Problems 12 and 13 suggest a method for finding the height of a building on a sunny day. In the diagram, the building will cast a shadow equal in length to AE. A stick of known length, held in the position shown, will cast a shadow equal to AD.



a. Why does $\angle ADB = \angle AEC$?

b. Why does $\angle DAB$ equal $\angle EAC$?

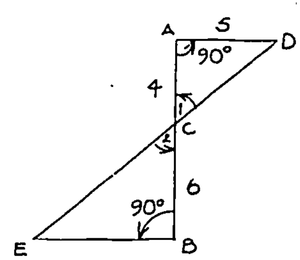
c. Why is ΔADB similar to ΔAEC?

d. Since ΔADB is similar to ΔAEC, the corresponding sides are (omitted word).

e. If BD = 1 meter, AD = 2 meters, AE = 40 meters, then the height of the building, CE = ?

15. Find the height of a building that casts a shadow of 27 meters at the same time a 2-meter stick casts a shadow of 3 meters.

16. Given the figure at the right, with $\angle A$ and $\angle B$ right angles and $\angle 1 = \angle 2$,



a. Does $\angle A = \angle B$?

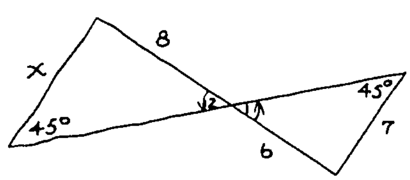
b. Why is ΔADC similar to ΔEBC?

c. What side in ΔADB corresponds to side EB in triangle EBC?

d. $\frac{5}{EB} = \frac{4}{?}$

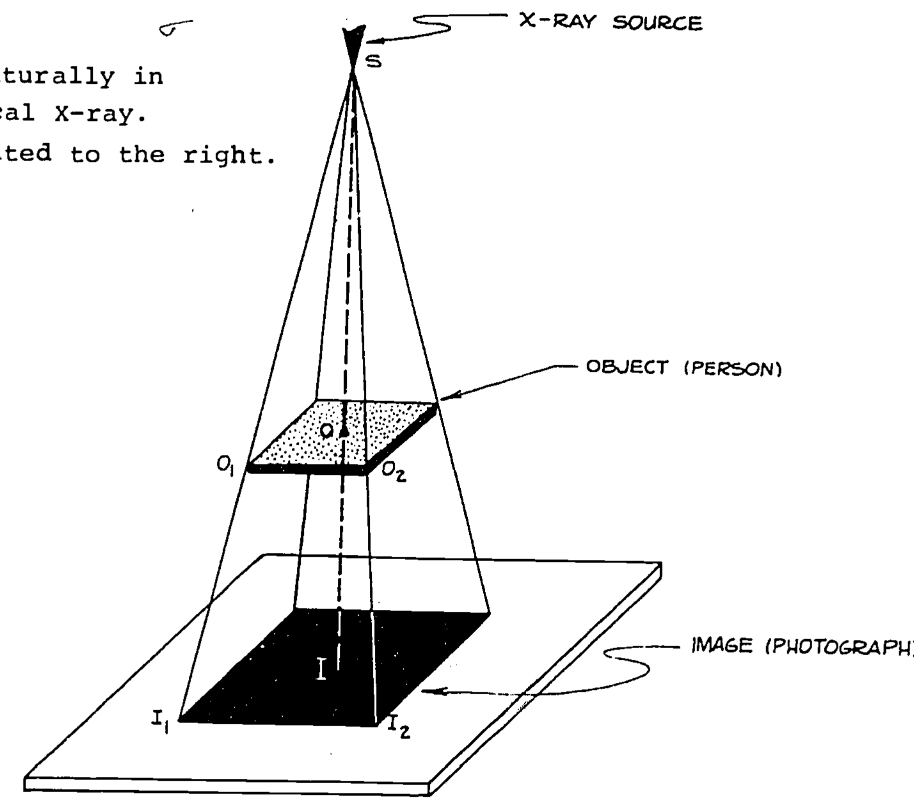
e. EB = ?

17. $\angle 1 = \angle 2$. Find x.



72

18. Magnification occurs naturally in the course of taking a medical X-ray. Why this happens is illustrated to the right.

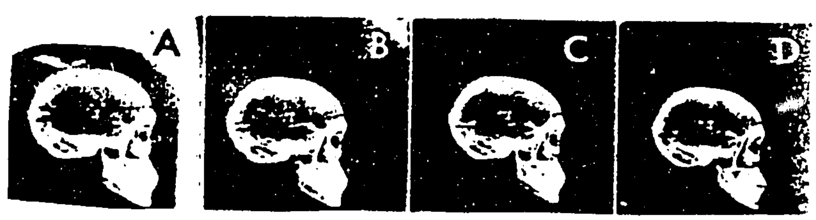


Note that the rays of light form similar triangles, i.e., triangle SO_1O_2 is similar to ΔSI_1I_2. An X-ray technician, for example, would be expected to be able to predict the degree of magnification for given values of object distance, SO, and image distance, SI. Since the image will always be farther away from the source than the object, the ratio

$$\frac{\text{length } I_1I_2}{\text{length } O_1O_2}$$

will always be greater than one. In other words, the image of the photographic plate will always be larger than the object. (Of course, all of the X-rays reproduced on this page have been reduced for obvious reasons.)

a. Examine the set of X-ray photographs of the same skull. In this set of photographs, the object-to-image distance OI was held constant and the source-to-image distance SI was varied. Notice that there is a variation in the size of the image.



SI = 40 cm 60 cm 80 cm 144 cm

It appears that a greater source to image distance (increases, decreases) the magnification.

b. Examine the two X-ray photographs on the following page. Only the distance of the skull from the photographic plate has been varied.

7.

SI =	40 cm	40 cm
OI =	8 cm	16 cm

Make a statement similar to the one in Part a about the effect of the object to image distance on the magnification.

c. Use similarity principles to determine the values of the blanks in the table. All distances are in centimeters.

	SI	SO	I_1I_2	O_1O_2
(1)	40	32		12
(2)	80	72		12
(3)	40		20	12
(4)	144	128	$13\frac{1}{2}$	

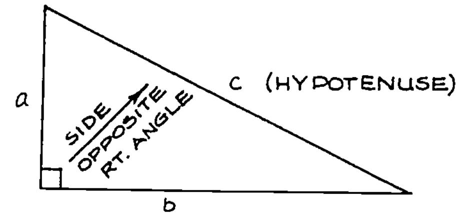
d. What is the object-to-image distance (OI) for (1) and (2)?

e. What is the object-to-image distance (OI) for (3) and (4)?

SECTION 13: THE PYTHAGOREAN THEOREM AND ITS CONVERSE

13-1 The Pythagorean Theorem

Trigonometric functions describe the relationships between the lengths of the sides of right triangles. (A right triangle is one that contains a right angle; the right angle is indicated by a small square.) In this section we will review one of the important properties of right triangles.



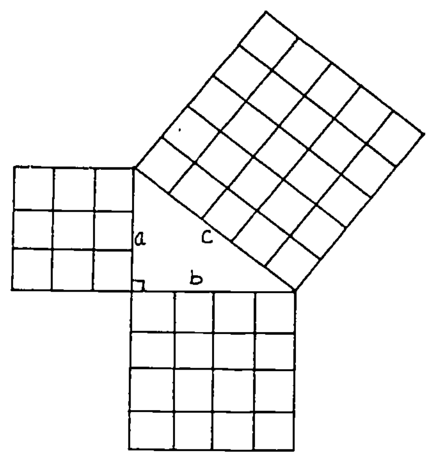
The side opposite the right angle of a right triangle is the hypotenuse. The hypotenuse is always the longest side of a right triangle.

The Pythagorean Theorem states that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. If the length of the hypotenuse is c and the lengths of the other sides are a and b,

$$a^2 + b^2 = c^2$$

For example, the triangle that has sides of 3, 4 and 5 is a right triangle. The square of 3 is 9; the square of 4 is 16. The square of the hypotenuse 5 is 25, which is equal to the sum of the squares of the other two sides.

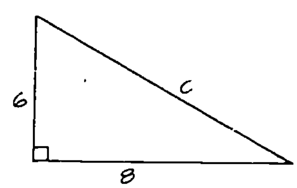
$$9 + 16 = 25$$



If the lengths of two sides of a right triangle are known, the length of the third side can be determined by means of the Pythagorean Theorem. The following examples illustrate the procedure.

EXAMPLE:

Determine the hypotenuse, c, in the triangle to the right.



SOLUTION:

$a = 6$

$b = 8$

$$c^2 = a^2 + b^2$$
$$c^2 = a^2 + b^2$$
$$c^2 = 6^2 + 8^2$$
$$= 36 + 64$$
$$= 100$$
$$c = \pm\sqrt{100}$$
$$= \pm 10$$

So, we see that algebraically c may be either positive or negative. However, since c is a length, the negative solution makes no sense. Therefore, c = 10.

EXAMPLE:

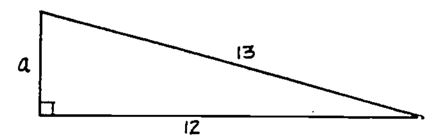
Determine side a in the triangle on the right.

SOLUTION:

$b = 12$ $\qquad$ $a^2 + b^2 = c^2$

$c = 13$ $\qquad$ $a^2 + 12^2 = 13^2$

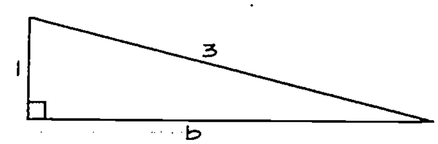
$$a^2 = 13^2 - 12^2$$
$$= 169 - 144$$
$$= 25$$
$$a = \sqrt{25}$$
$$= 5$$



All three sides in our two examples were integers. When one or more of the sides is not an integer, you may have an opportunity to use the techniques for manipulating radicals that you learned in Unit III, Section 6-3. The next example involves simplifying a radical expression.

EXAMPLE:

Determine side b.



SOLUTION:

$$b^2 = 3^2 - 1^2$$
$$= 9 - 1$$
$$= 8$$
$$b = \sqrt{8}$$
$$= \sqrt{2 \cdot 4}$$
$$= 2\sqrt{2}$$

(Since $4 = 2^2$, the 4 may be removed from under the radical sign and replaced with a factor of 2 in front of the radical.)

13-2 Rationalizing a Denominator (A Review)

Expressions in which a radical appears in a denominator are difficult to work with. For this reason, radicals are usually removed from denominators. Removing a radical from a denominator is called rationalizing the denominator.

The procedure for rationalizing a denominator uses the fact that

$$\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$$

The denominator of the expression $\frac{1}{\sqrt{2}}$ is rationalized by multiplying both numerator and denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$
$$= \frac{\sqrt{2}}{2}$$

This operation is possible because $\frac{\sqrt{2}}{\sqrt{2}} = 1$.

Observe that rationalization of this denominator makes calculations easier. A decimal approximation for $\sqrt{2}$ is 1.41421356... It is easier to divide 2 into 1.41421356... than to divide 1.41421356... into 1.

EXAMPLE: Rationalize the denominator of the expression $\frac{4}{\sqrt{3}}$.

SOLUTION:
$$\frac{4}{\sqrt{3}} = \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$$
$$= \frac{4\sqrt{3}}{3}$$

EXAMPLE: Rationalize the denominator of the expression $\frac{12}{\sqrt{6}}$.

SOLUTION:
$$\frac{12}{\sqrt{6}} = \frac{12}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}}$$
$$= \frac{12\sqrt{6}}{6}$$ (This expression may be further simplified by dividing 12 by 6.)

$$\frac{12\sqrt{6}}{6} = 2\sqrt{6}$$

In Section 6-3 of Unit III denominators were rationalized. However, the denominator was always a power of ten. This time, that simplifying feature will be removed.

13-3 A No-No

We must warn one operation that is not permitted: $\sqrt{a^2 + b^2}$ is never equal to $\sqrt{a^2} + \sqrt{b^2}$ unless a or b = 0. This may be illustrated by setting a = 3 and b = 4.

$$\sqrt{3^2 + 4^2} = \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

Now we evaluate $\sqrt{3^2} + \sqrt{4^2}$ to demonstrate that

$$\sqrt{3^2 + 4^2} \neq \sqrt{3^2} + \sqrt{4^2}$$

We know that the left side is 5. Therefore,

$$5 \neq \sqrt{3^2} + \sqrt{4^2}$$

Since $\sqrt{3^2} = 3$ and $\sqrt{4^2} = 4$,

$$5 \neq 3 + 4$$
$$5 \neq 7$$

and we confirm the inequality

$$\sqrt{a^2 + b^2} \neq \sqrt{a^2} + \sqrt{b^2}$$

for the special case when a = 3 and b = 4.

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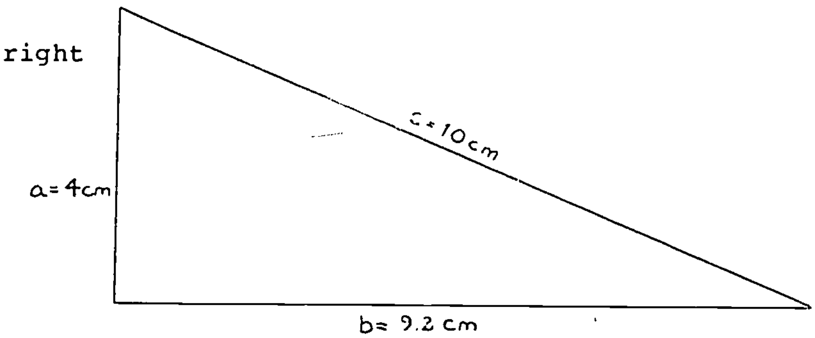
13-4 Using the Pythagorean Theorem to Test for Right Triangles

The converse of the Pythagorean Theorem is also true: if the relationship $a^2 + b^2 = c^2$ is true for the sides a, b and c of a triangle, the triangle is a right triangle. This fact may be used to determine whether a particular triangle is a right triangle. The next two examples illustrate the procedure.

EXAMPLE:

Is the triangle at the right a right triangle?



SOLUTION:

If the triangle is a right triangle, $a^2 + b^2 = c^2$.

$a = 4$

$b = 9.2$

$c = 10$

$$\begin{aligned} a^2 + b^2 &= 4^2 + 9.2^2 \\ &= 16 + 84.64 \\ &= 100.64 \\ c^2 &= 10^2 \\ &= 100 \end{aligned}$$

c^2 is equal to 100, while $a^2 + b^2$ is equal to 100.64

$$a^2 + b^2 \neq c^2$$

Therefore, the triangle is not a right triangle, even though it looks like one. (You can measure the sides to confirm that the figure is an accurate representation.)

EXAMPLE:

Is the triangle at the right a right triangle?

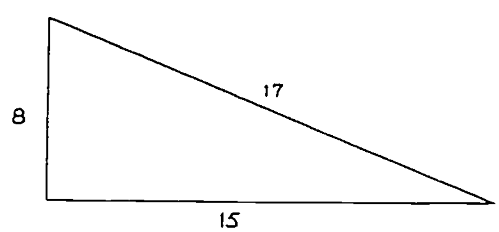
SOLUTION:

$a = 8$

$b = 15$

$c = 17$

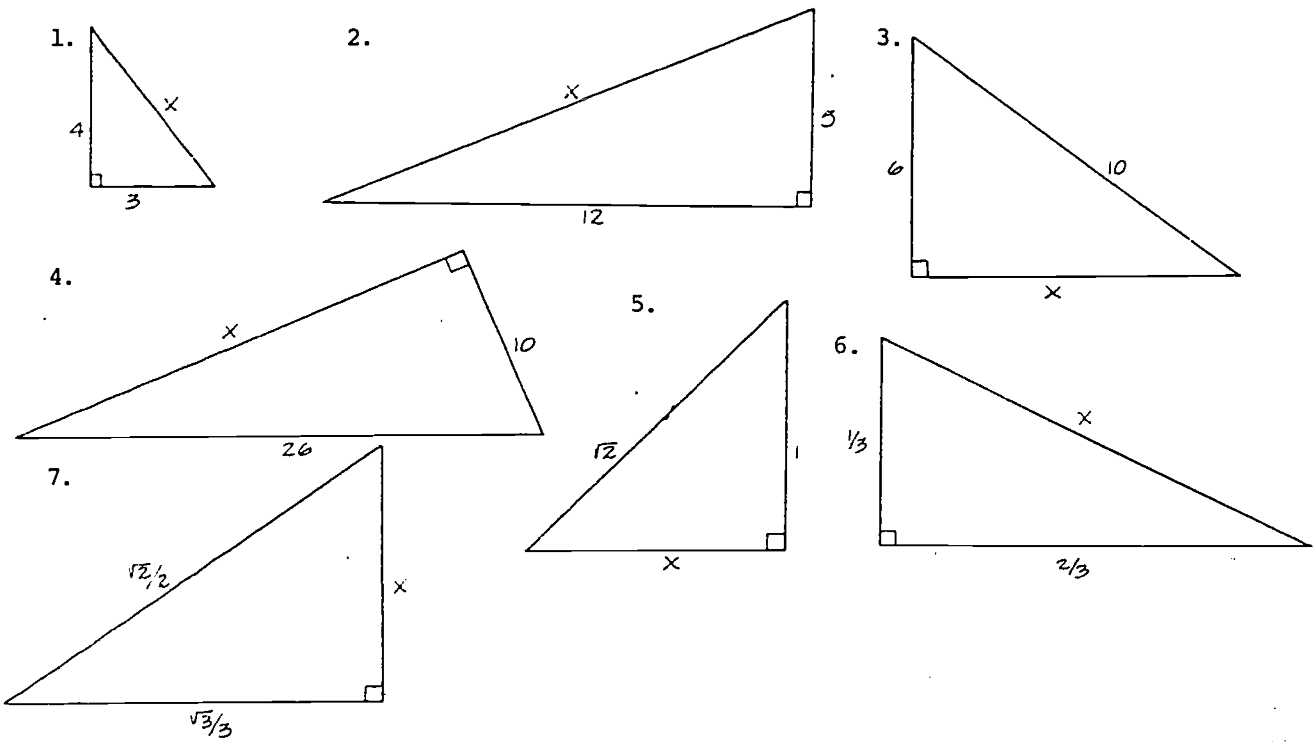
$$\begin{aligned} a^2 + b^2 &= 8^2 + 15^2 \\ &= 64 + 225 \\ &= 289 \\ c^2 &= 17^2 \\ &= 289 \end{aligned}$$



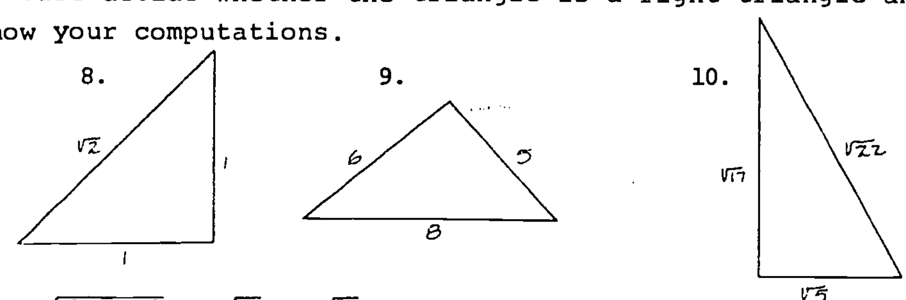
c^2 is equal to 289, and $a^2 + b^2$ is also equal to 289. Since $a^2 + b^2 = c^2$, the triangle is a right triangle.

PROBLEM SET 13:

Use the Pythagorean Theorem to find x in the following problems. Remember to simplify and rationalize the denominators of answers containing radicals.



In each case decide whether the triangle is a right triangle and write "yes" or "no." Show your computations.



11. a. Does $\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2}$?

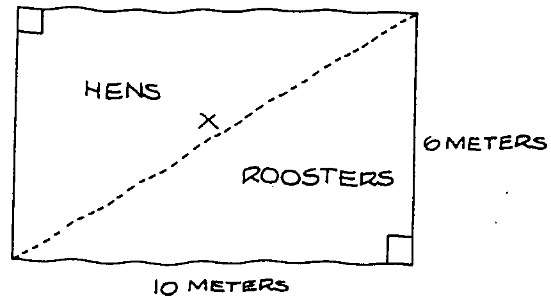
b. Let a = 7 and b = 24, for example. Does $\sqrt{7^2 + 24^2} = \sqrt{7^2} + \sqrt{24^2}$? Demonstrate your answer.

12. A rectangular field is 30 meters long and 40 meters wide. How long is the diagonal?

13. Can a 87-cm cane be placed flat on the bottom of a trunk which is 76 cm long and 41 cm wide? Prove your answer.

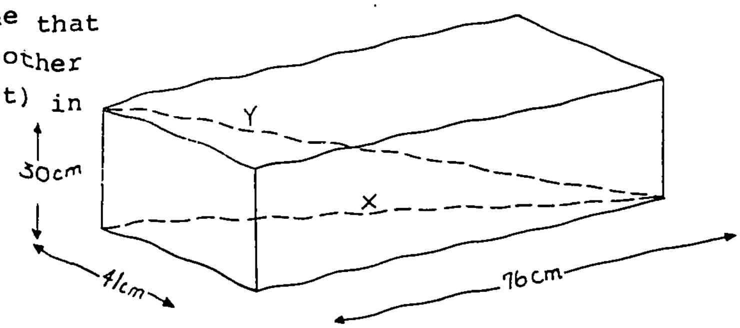
*14. Two cars leave a city, one traveling east at 30 kilometers per hour and the other south at 40 kilometers per hour. How far apart are the cars in 2 hours?

15. Elmo has taken up farming. He has a rectangular chicken pen which is 10 meters by 6 meters. He wishes to divide the pen into two compartments, one for hens and one for roosters. He plans to do this by putting chicken wire along the diagonal as shown at right.

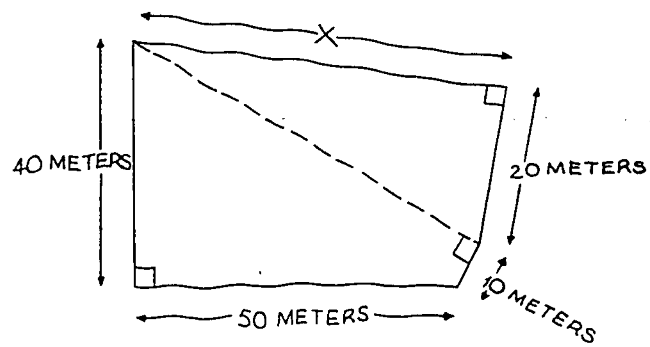


How many meters of chicken wire will Elmo need? Express your answer as a decimal. Use the square root table of Unit III, pp. 91-94.

*16. A trunk is 76 cm long, 41 cm wide and 30 cm high. What is the longest cane that may be placed inside the trunk? In other words, find length y (figure at right) in terms of a decimal number.

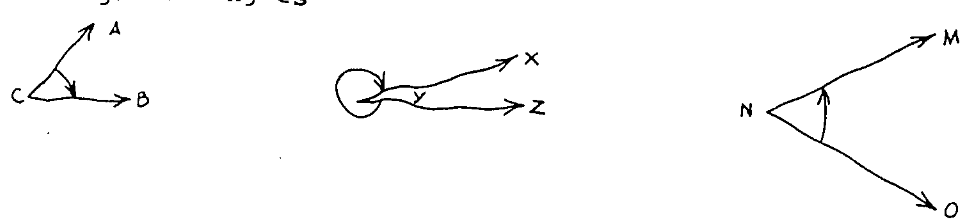


*17. Norbert Numnitz has purchased a lot with a peculiar shape. His map has all the dimensions and angles as shown in the diagram to the right, but neglects to give x. What is the length of x in meters?

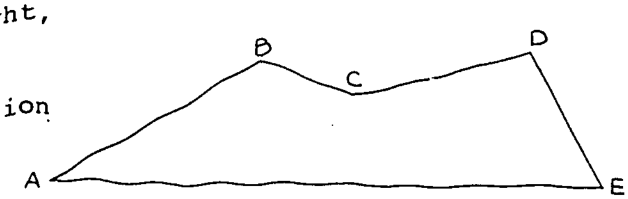


REVIEW PROBLEM SET 14:

1. Identify the negative angles.



2. True or False. (Refer to diagram at right, and remember to rotate counterclockwise.)



a. $\angle ABC = \angle ABC$

b. $\angle ABC = \angle CBA$

c. $\angle CDE = \angle EDC$

d. $\angle AED < 180°$

e. $\angle BAE < \frac{1}{4}$ revolution

f. $\angle BCD < \pi$ radians

g. $\angle CDE < \frac{1}{2}$ cycle

h. $90° < \angle ABC < 180°$

3. Convert the following to radian measure.

a. 180° b. 360° c. 60° d. 15° e. 420°

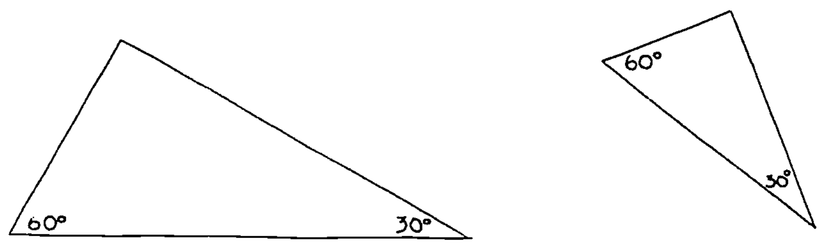
4. Convert to degree measure.

a. $\frac{\pi}{9}$ radians
b. $\frac{\pi}{180}$ radians
c. $\frac{3\pi}{2}$ radians
d. 8π radians
e. .5 revolution
f. .8 revolution
g. 2.3 cycles

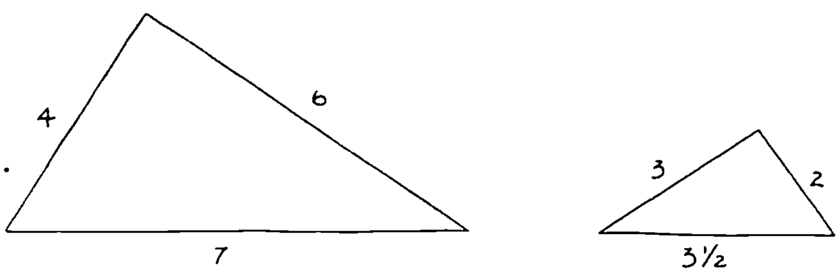
5. If you walk 60π meters along the circumference of a circle with radius 12 meters, you have formed what angle at the center of the circle? State your answer in: (a) radians, (b) degrees, (c) revolutions, (d) cycles.

6. In each of the following decide whether the two triangles are similar. If they are not, write "ns" and explain why. If they are similar, write AA, SAS or SSS to indicate the theorem on similar triangles you used.

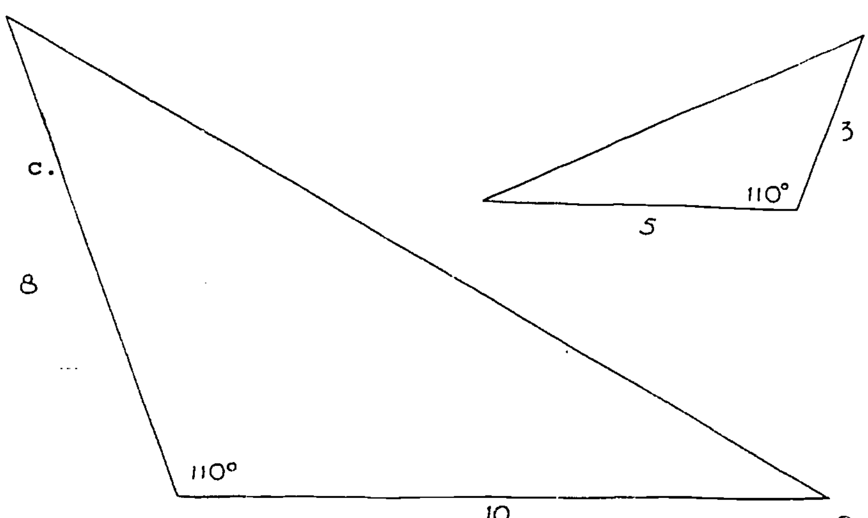
a.



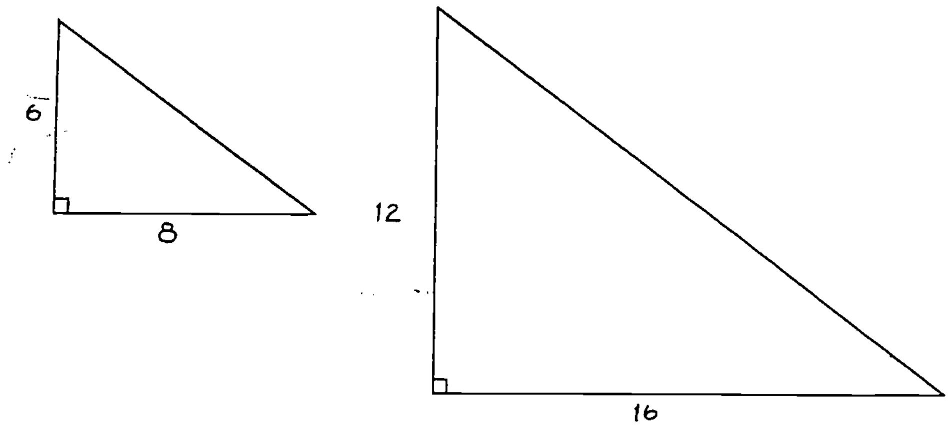
b.



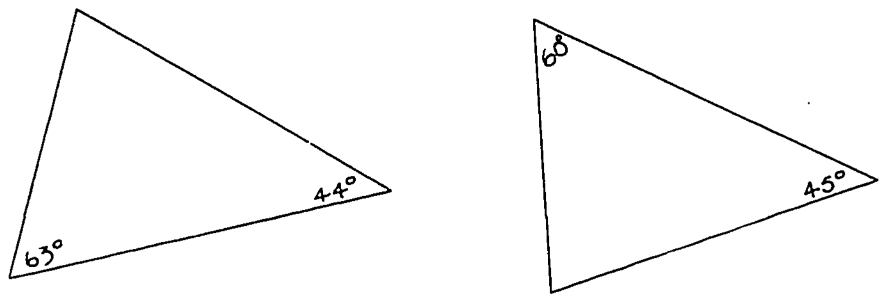
c.



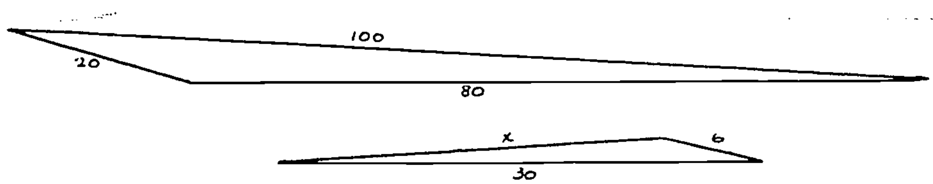
d.



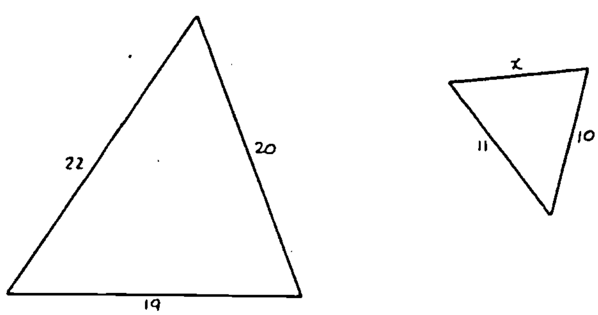
e.



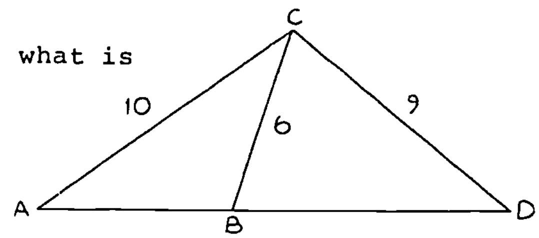
7. Given that the triangles below are similar, what is x?



8. Given that the triangles below are similar, what is x?



*9. Given that ΔABC is similar to ACD, what is the length of side AD?



10. Bruno wants to paint a lifesize picture of his favorite wrestler, Rabid Runamok, on his living room wall, using as a guide a small snapshot of the wrestler. Bruno recalls that in real life Rabid stands 214 cm tall. Bruno notes that the height of Rabid's image in the snapshot is 13 cm. He also finds that the distance between the wrestler's biceps in the snapshot is 5 cm. In the wall painting what should the distance between the biceps be? Round to the nearest whole centimeter.

11. Simplify the following radical expressions by removing all perfect square factors.

a. $\sqrt{8}$ c. $\sqrt{700}$ e. $\sqrt{27a^4}$ g. $\sqrt{10^8}$

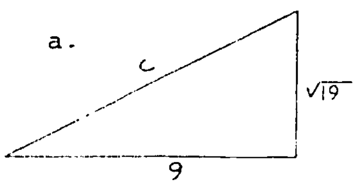
b. $\sqrt{121}$ d. $\sqrt{99}$ f. $\sqrt{100a^7}$

12. In the following problems, remove all perfect square factors and rationalize all denominators.

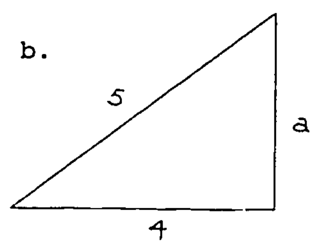
a. $\frac{1}{\sqrt{11}}$ b. $\frac{2}{\sqrt{7}}$ c. $\frac{10}{\sqrt{8}}$ d. $\frac{2}{\sqrt{12}}$

13. Use the Pythagorean Theorem to find the length of the unknown side.

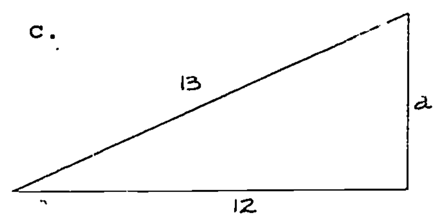
a.



b.

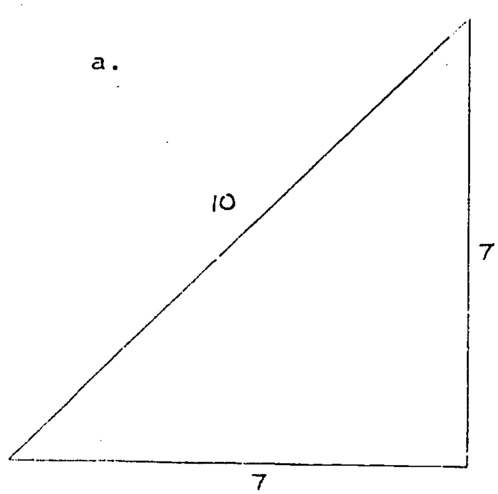


c.

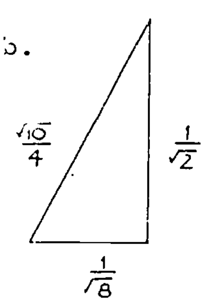


14. In the following, decide whether the given triangle is a right triangle. Show your work.

a.



b.

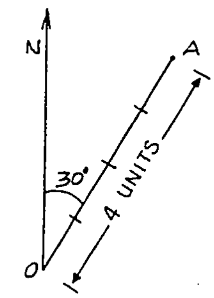


8.

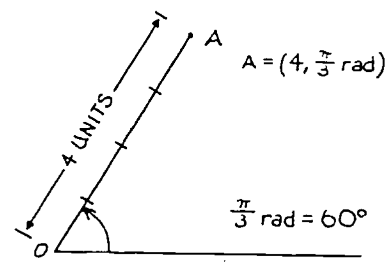
SECTION 15: POLAR COORDINATES AND POLAR VECTORS

15-1 Polar Coordinates

Up until now we have always stated ' cation of a point in a plane in terms of two rectangular coordinates. Th was the system of Cartesian coordinates, in which a point is located in relation to two perpendicular lines, conventionally know as the x-axis and y-axis. All subsequent location of points has been in te f x- and y-coordinates.

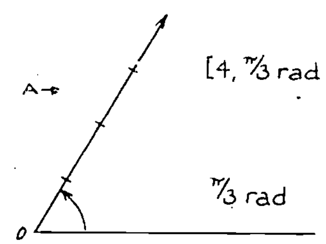


Another sy... that may be used, now that you have some experience with angles, is one in which a location is specified by the distance from a reference point (the origin) and the measure of the angle of rotation starting from a reference line. Point A in the figure at right is 4 units from the origin O in a direction 30 degrees east of north. This system is used in navigation.

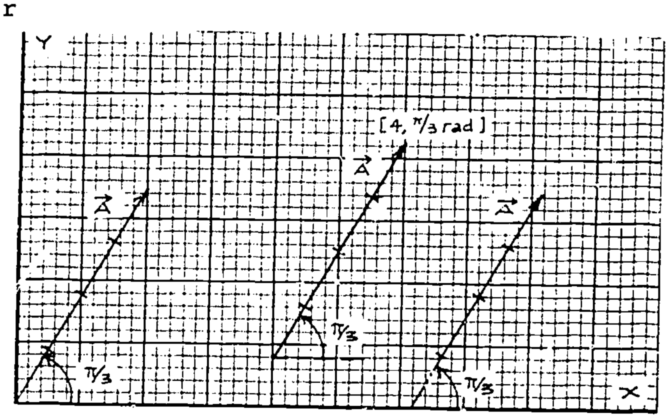


In the realms of mathematics and the sciences, however, other conventions are used. The directions are related not to a line drawn north from the origin but to a horizontal line drawn to the right of the origin, as the x-axis is. Thus we may specify the location of A as (4, $\frac{\pi}{3}$ rad). This is the address of A in <u>polar coordinates</u>. The first coordinate given means that the distance of A from the origin is 4 units. The second coordinate indicates that the angle between line OA and the horizontal axis is $\frac{\pi}{3}$ radians or 60°.

15-2 Polar Vectors



We have until now expressed two-component <u>vectors</u> in terms of an x-component and a y-component. However, we may also express a vector in terms of polar coordinates. The first component of a vector in polar coordinates is its length; the second component is its direction in terms of its clockwise or counterclockwise rotation from the x-axis. Therefore we specify the vector at the right as [4, $\frac{\pi}{3}$ rad]. The square brackets mean that we are talking about a vector instead of a pair of coordinates.

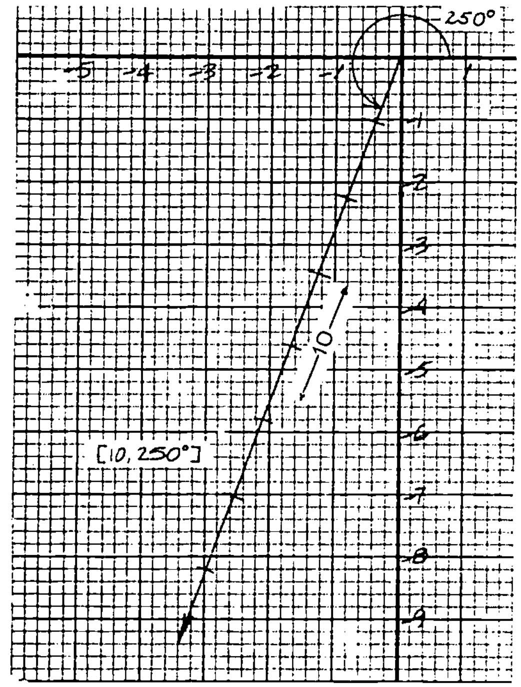


The designation [4, $\frac{\pi}{3}$ rad] may refer not only to $\vec{A}$ but to any vector equivalent to $\vec{A}$. All of the vectors in the graph at right may be represented as $\vec{A}$ = [4, $\frac{\pi}{3}$ rad].

The length of a polar vector is its magnitude. The magnitude of $\vec{A}$ is 4. The angle of a polar vector is its direction. The direction of $\vec{A}$ is $\frac{\pi}{3}$ radians.

15-3 Converting Between Rectangular and Polar Coordinates

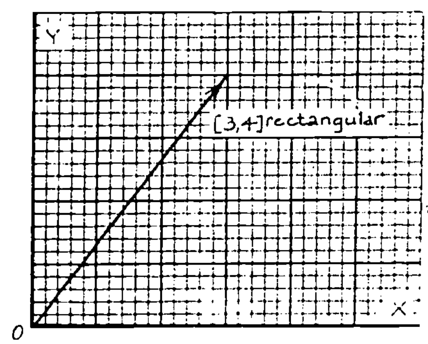
A vector stated in polar terms may be thought of as the sum of two vectors, one equal to the x-component of the vector and the other equal to the y-component. Consider the vector [10, 250°] shown in the graph to the right.



We may read the x- and y-coordinates directly. They are approximately -3.4 and -9.4.

Thus the vector which is [10, 250°] in polar coordinates is about [-3.4, -9.4] in rectangular coordinates.

On the other hand, a vector that is stated in Cartesian coordinates may also be stated in polar coordinates. Consider the vector whose x- and y-components are given by [3, 4]. The vector is drawn from the origin to the terminal point (3, 4). (See second graph at right.) To express this vector in polar coordinates we must determine its magnitude and direction. Its magnitude may be found by measuring it on the graph, or it may be found using the Pythagorean Theorem.

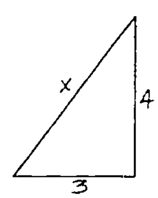


$$x^2 = 3^2 + 4^2$$

$$= 9 + 16$$

$$= 25$$

We take the square root of both sides and obtain



$$\sqrt{x^2} = \sqrt{25}$$

$$x = 5$$

The magnitude of the vector is 5.

To find the angle of the vector, we may measure it on the graph using a protractor. The angle between the x-axis and the vector is approximately 53 degrees. Therefore, the vector is given in polar coordinates as [5, 53°]. When the angle is expressed in terms of radians the vector is [5, $\frac{53\pi}{180}$ rad] or approximately [5, .925 rad].

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PROBLEM SET 15:

1. Sketch the following vectors on graph paper. Use a protractor and a ruler. (Convert the radian measure to degrees first.)

$\vec{A} = [4, \frac{\pi}{6} \text{ rad}]$ $\vec{B} = [5, \frac{5\pi}{2} \text{ rad}]$ $\vec{C} = [8, \frac{4\pi}{3} \text{ rad}]$

2. a. Plot the point (5, 4).

b. Draw a vector from the origin to the terminal point (5, 4).

c. Use the Pythagorean Theorem to determine the magnitude of the vector.

d. Determine the approximate direction of the vector using a protractor.

e. The approximate polar representation of the vector is [?, ?]. (State the angle measure in degrees.)

3. Convert the following pairs of rectangular coordinates to polar coordina[illegible] (Plot the points on rectangular coordinate graph paper. Then use a protractor and the Pythagorean Theorem.)

a. (1, 1) c. (-5, 1) e. (-?, -4)

b. (-1, 1) d. (3, 0)

4. a. Sketch the following vectors on graph paper.

(1) [-2, 2] (2) [-6, -1] (3) [3, -2]

b. Determine the approximate polar equivalents of the vectors in Part a.

5. Convert the following pairs of polar coordinates to their approximate rectangular equivalents. (Use a protractor and centimeter ruler to plot the points on rectangular coordinate paper. Then read the approximate x- and y-coordinates.)

a. (3, 30°) b. (4, 450°) c. (2, 150°) d. (4, 660°)

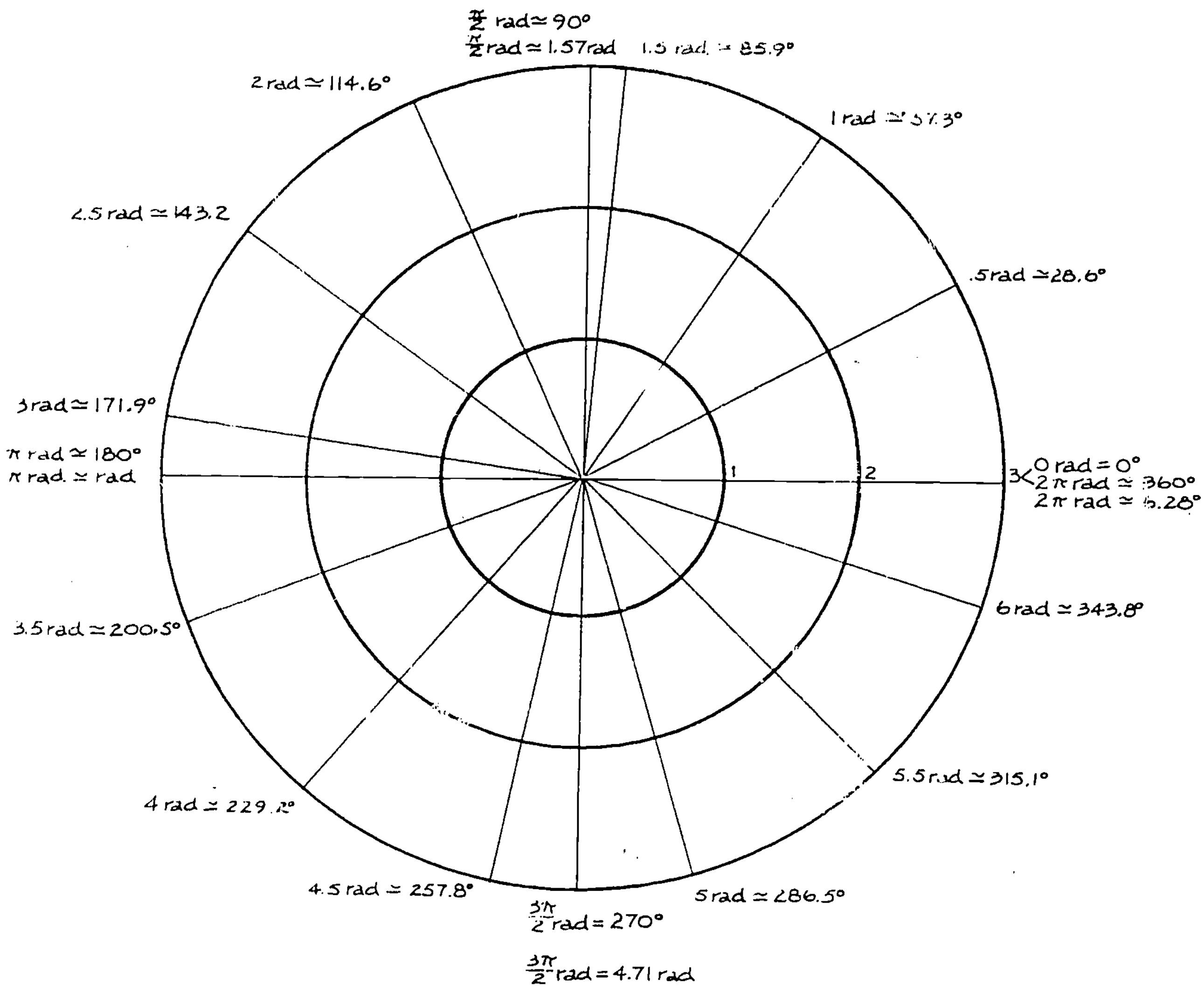
6. Using a compass, ruler and protractor, construct a figure similar to the one shown on the following page. Such figures are called "polar graphs."

7. On the polar graph that you have constructed, locate the following points. All coordinates are polar. <u>Label all points</u>.

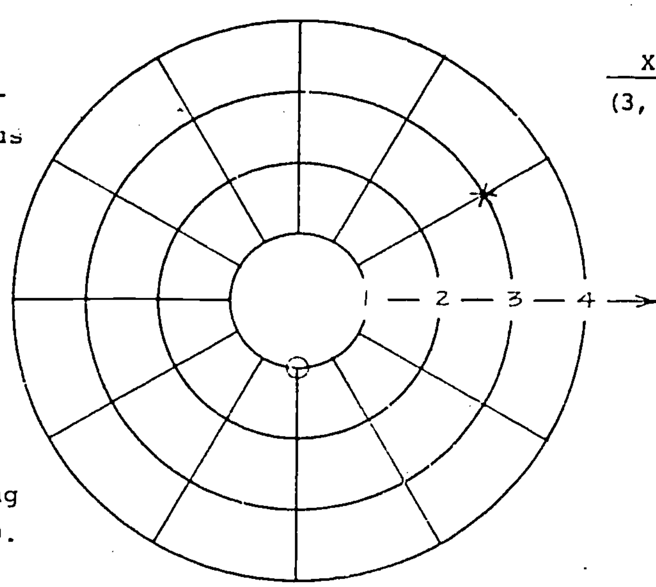
a. (3, 2 rad) b. (2, 4.5 rad) c. (1, 3.14 rad) d. (2, [illegible] rad)

8. Graph the following polar vectors on the polar graph.

$\vec{A} = [2, 2.5 \text{ rad}]$ $\vec{B} = [1, 6 \text{ rad}]$ $\vec{C} = [3, 4.71 \text{ rad}]$ $\vec{D} = [3, .5 \text{ rad}]$

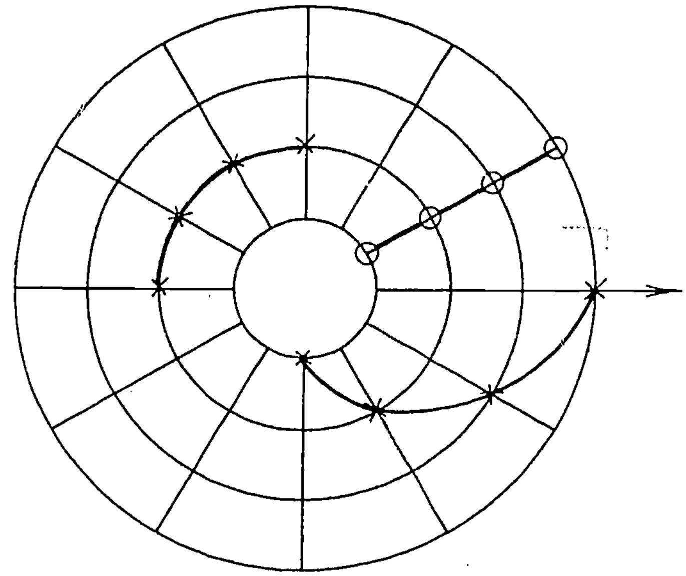


The next questions relate to the game of polar tic-tac-toe. This game is played on a polar-coordinate system with circles of radius 1, 2, 3 and 4 and radial lines every $\frac{\pi}{6}$ radians (30°). The origin is omitted because of the disproportionate advantage it gives one player. We can describe the progress of a game by listing the polar ordered pairs corresponding to the positions played. At right is the beginning of a game between players X and O. (The abbreviation "rad" has been omitted.)



X	O
$(3, \frac{\pi}{6})$	$(1, \frac{3\pi}{2})$

In order to win a game of polar tic-tac-toe, a player must place four marks in a row along a circle, radius or diagonal spiral. The three combinations shown at right are all winners.



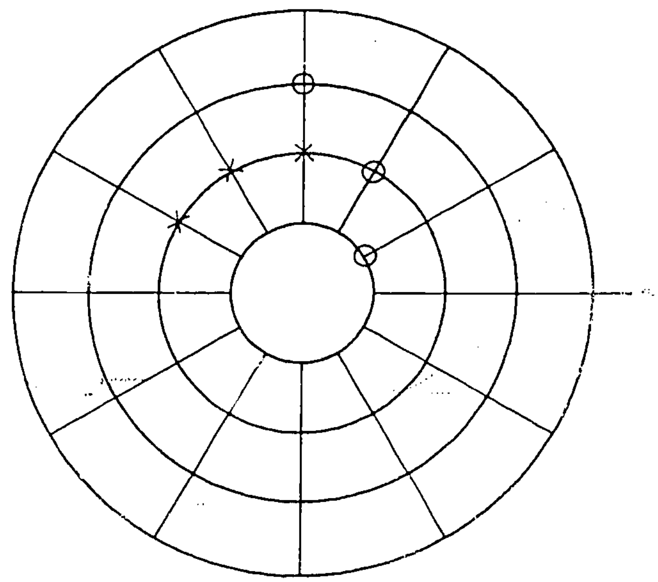
9. The diagram at right shows the beginning of a game.

a. List the positions played by X as polar ordered pairs (express all angles in radians).

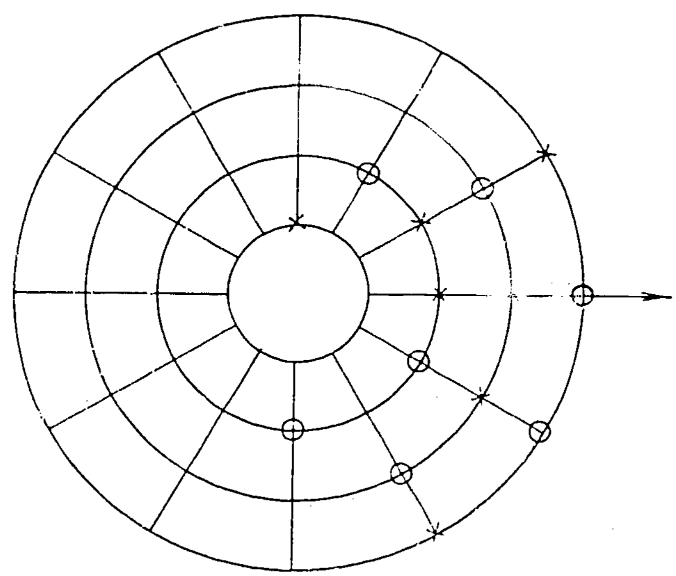
b. List the positions played by O.

c. If X moves next, what position must she play to win?

d. If O moves next, what position must he play to win?



10. The diagram at right shows part of a game. If X is to play next, what polar point should she choose in order to win?



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11. At right is a list of moves in part of a game.

a. Draw a polar coordinate system and plot the positions listed at right.

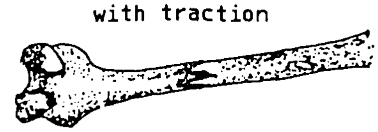
*b. It is O's turn. O can win in two moves if he makes the correct move. What position should he play next?

X	O
$(3, \pi)$	$(3, \frac{5\pi}{6})$
$(2, \pi)$	$(4, \pi)$
$(2, \frac{2\pi}{3})$	$(2, \frac{5\pi}{6})$
$(1, \frac{5\pi}{6})$	$(3, \frac{\pi}{2})$
$(3, \frac{7\pi}{6})$	$(4, \frac{4\pi}{3})$
$(2, \frac{7\pi}{6})$	

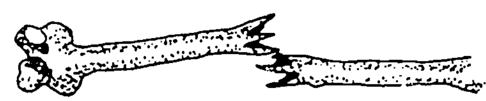
SECTION 16: POLAR VECTORS AND FORCES

Forces are most conveniently represented by vectors. Forces come in all possible sizes and may come from any direction. When an object is acted upon by two or more forces, the vector sum of the forces may be found by the method of vector addition--the same kind of vector addition that you studied earlier in the context of nutrition.

with traction



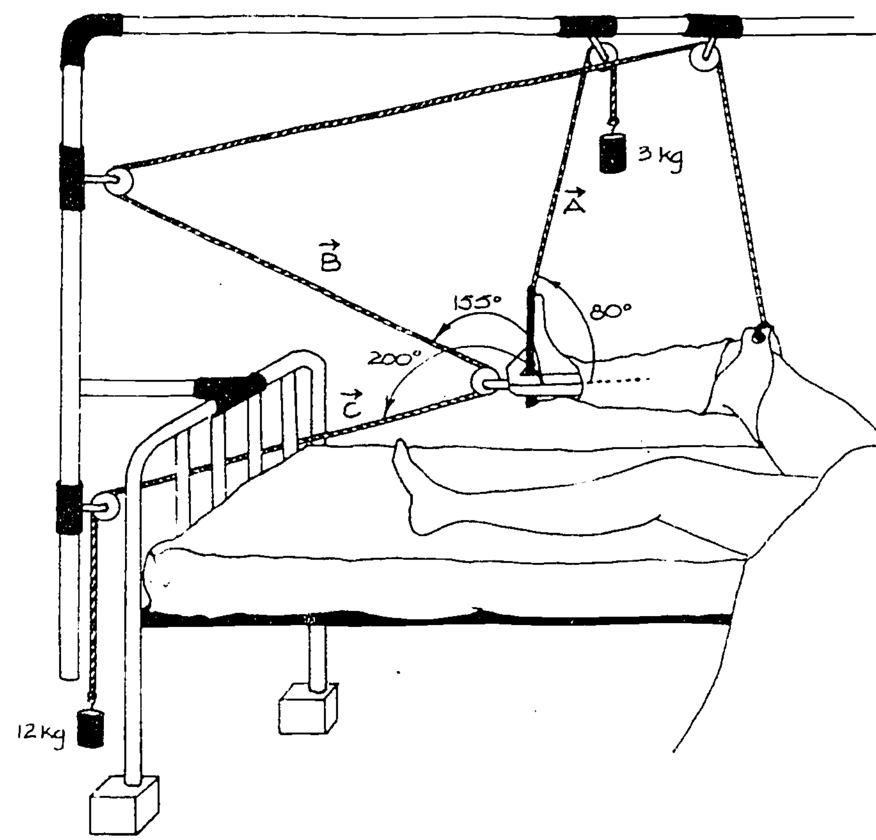
without traction



EXAMPLE:

Elmira broke her leg in a motorcycle accident. Consequently, she had to spend some time in traction. The objective of traction is to keep the two ends of a fractured bone just touching. Without traction the contraction of the muscles can pull the two ends out of alignment

A typical arrangement is shown at the right.



We may use vector addition to answer the question, "What is the vector sum of the forces acting on the foot?" Skills similar to those to be demonstrated may be required of a health care worker in the field.

SOLUTION:

First we determine the polar representation of each of the three forces.

Force A: The magnitude of this force is 3 kg. It is the force produced by the 3-kg weight and transmitted to the foot by way of the rope and pulley. The direction of this vector is 80° as shown. Therefore,

$$\vec{A} = [3\text{ kg}, 80°]$$

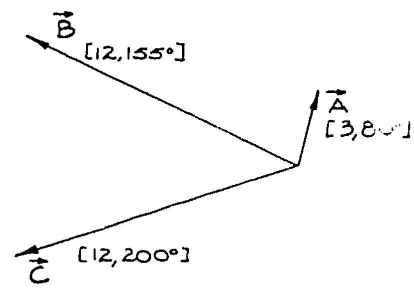
Force B: The magnitude of this vector is 12 kg, the tension produced in the rope as a result of the 12-kg weight. Its direction is 155° as shown. Therefore,

$$\vec{B} = [12\text{ kg}, 155°]$$

Force C: The magnitude of this vector is 12 kg and its direction is 200° as shown. Therefore,

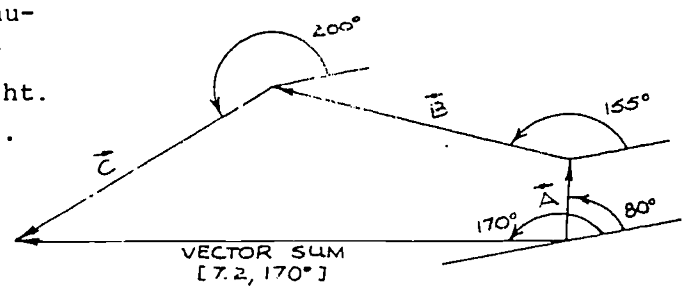
$$\vec{C} = [12\text{ kg}, 200°]$$

Second, we draw a diagram of the three vectors. Notice that $\vec{B}$ and $\vec{C}$ are four times as long as $\vec{A}$.



Third, we add the vectors, just as nutritional vectors were added, by placing the vectors head to tail as shown at right. Therefore, the vector sum is [7.2, 170°].

The vector sum of a set of forces is called the resultant. The magnitude and direction of the resultant may be found by simply measuring the length and direction of the vector sum. Later, we will use trigonometric functions to add force vectors. For the present, however, we will demonstrate the use of graphical methods.

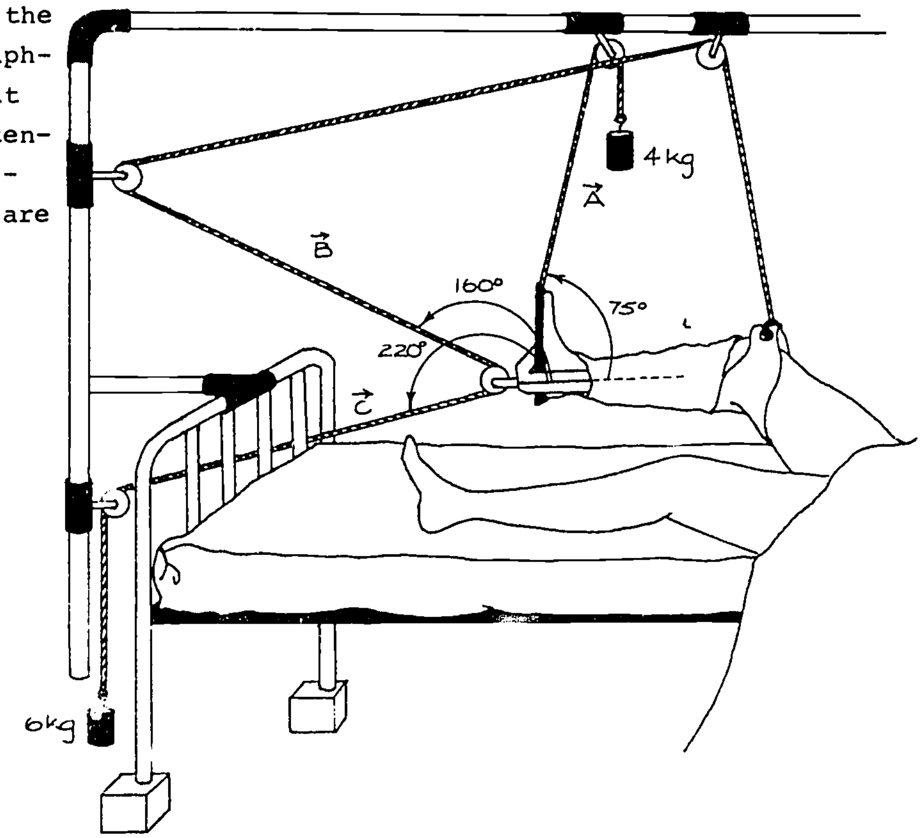


PROBLEM SET 16:

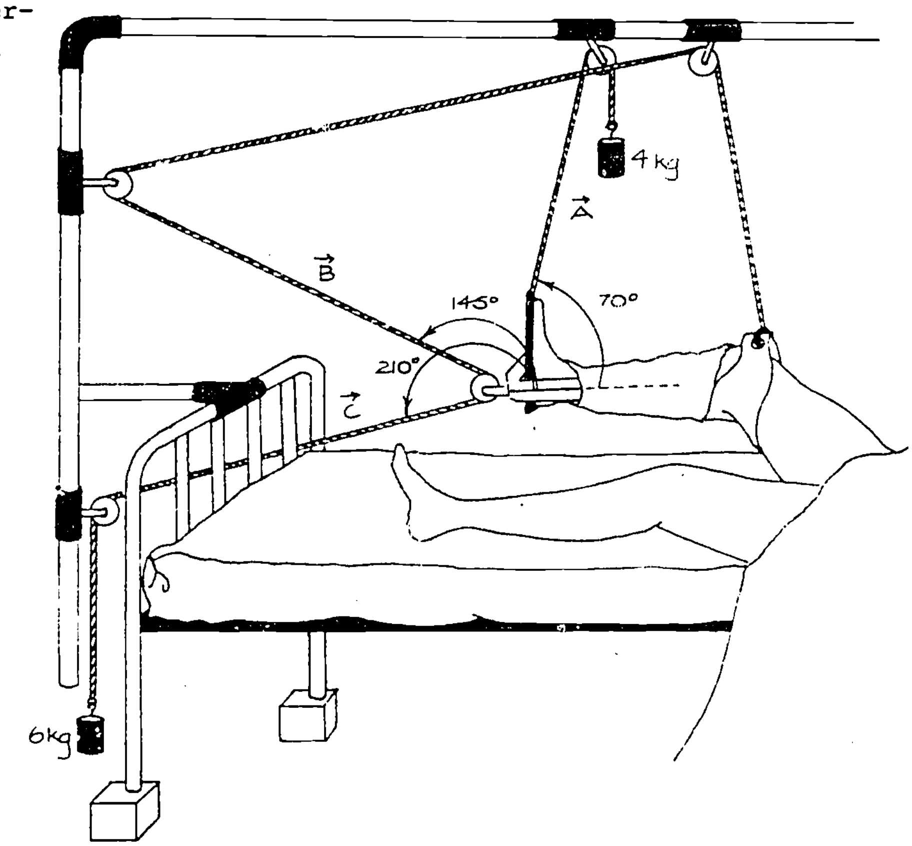
For Problems 1 through 6 find the vector sum by graphical methods. After adding the vectors, use a protractor and ruler to find the polar components of the resultant.

1. [3, 60°] + [5, 45°] = [?, ?]
2. [6, 45°] + [10, 120°] = ?
3. [2, 30°] + [4, 120°] = ?
4. [7, 50°] + [9, 280°] = ?
5. [5, 25°] + [6, 130°] = ?
6. [3, 10°] + [4, 170°] = ?

7. Determine the vector sum of the forces acting on the foot by graphical methods. The 6-kg weight at the foot of the bed produces a tension of 6 kg in the rope. Therefore, the magnitudes of $\vec{B}$ and $\vec{C}$ are both 6 kg.



8. Use graphical methods to determine the vector sum of the forces acting on the foot.



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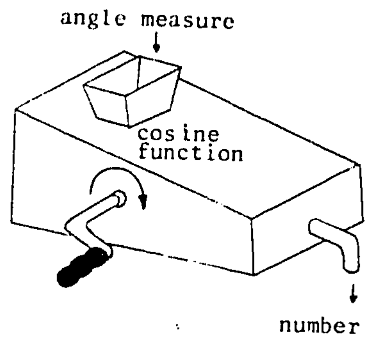
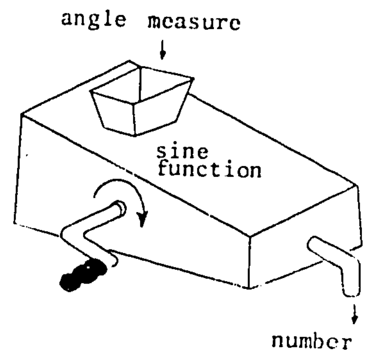
SECTION 17: COSINE AND SINE

17-1 Trigonometric Functions

Cosine and sine are trigonometric functions. The domain numbers or inputs for these functions are the measures of angles. The outputs produced by these functions are real numbers.





Thus we are adding a new type of function to those already studied, i.e., linear functions, quadratic functions, the chi-square function, and logic functions.

17-2 Cosine and Sine

The cosine and sine of the angle 30° are related to the rectangular components of the polar vector [1, 30°]. Examine Figure 1 below.

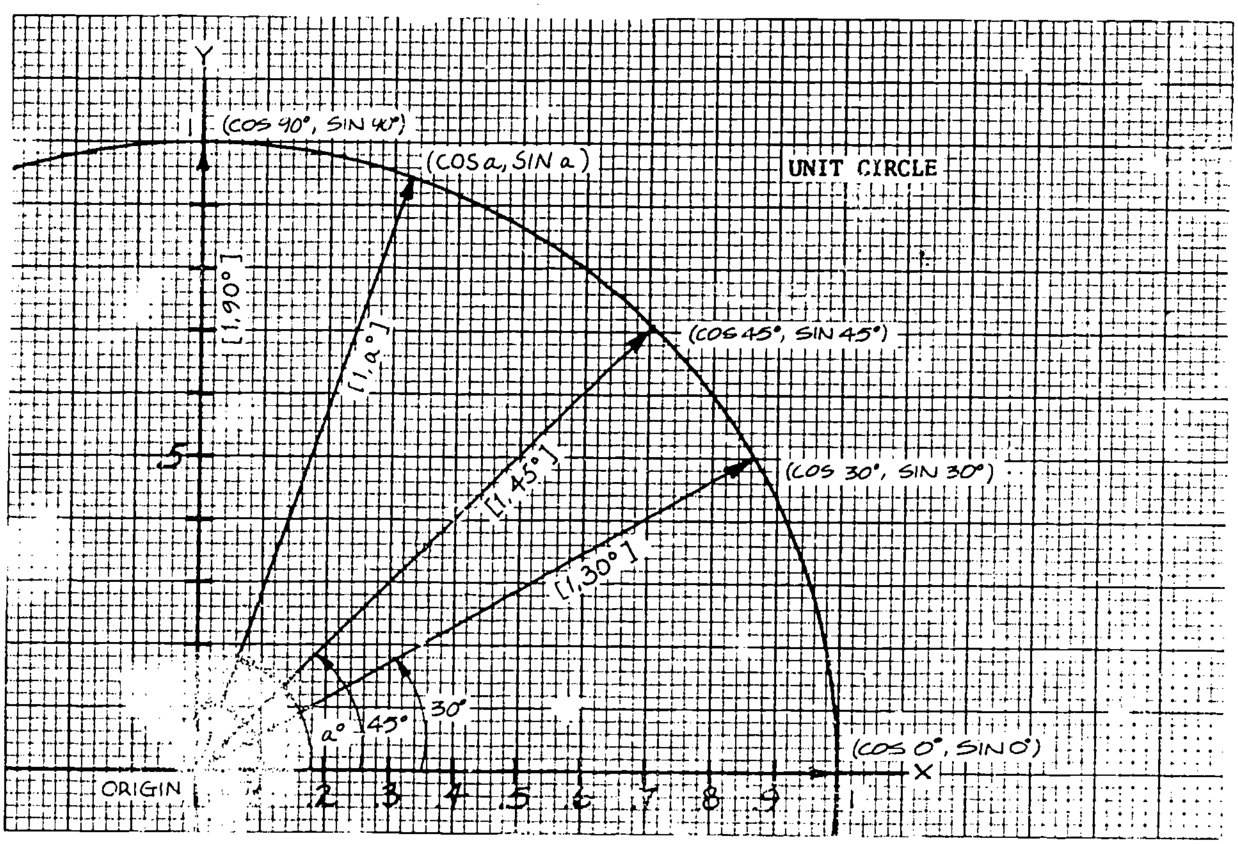


Figure 1

Now inspect the vector [1, 30°] at the point where the head of the vector touches the unit circle.

The cosine of 30° is the x-component of the vector [1, 30°].

The x-component of the polar vector [1, 30°] is determined from the graph to be approximately .87. Cosine is abbreviated "cos," so we may write

$$\cos 30° \approx .87$$

The sine of 30° is the y-component of the vector [1, 30°].

This coordinate is measured to be .50. Sine is abbreviated "sin," and we may write

$$\sin 30° = .50$$

In the same fashion, we may find cos 45° and sin 45°. Cos 45° is seen to be about .71. Likewise, sin 45° ≈ .71.

To find the cosine of any angle, we may proceed in the same fashion. We find the x-component of the vector [1, angle]. Likewise for sin (angle); it is the y-component of the vector [1, angle]. We let α (the Greek letter "alpha") stand for any angle and state the general definitions of cos α and sin α below for easy reference.

cos α is the x-component of the polar vector [1, α].

sin α is the y-component of the polar vector [1, α].

Given the above definitions, we may determine the cosines and sines of 0°-90°. Referring to the graph, we see that

$$\cos 0° = 1 \qquad \cos 90° = 0$$

$$\sin 0° = 0 \qquad \sin 90° = 1$$

17-3 The Signs of the Functions in the Four Quadrants

The x- and y-axes divide the unit circle into four regions called "quadrants." For angles in the first quadrant (see the region labeled "I" in Figure 2 on the next page) both cosine and sine are positive. This is not true for all quadrants. The signs of the x- and y-coordinates change in the different quadrants and therefore the signs of the sine and cosine change also.

Figure 2 shows the signs of the coordinates in each of the quadrants. For example, in quadrant II we find the symbols (-, +) which means that the x-coordinate is negative and the y-coordinate is positive. The vector [1, 160°] lies in quadrant II; therefore, cos 160° (the x-coordinate) is negative, as is cos 200°. However, cos 300° is positive because the x-coordinate is positive in quadrant IV.

The sine is positive in quadrants I and II because the vertical coordinates are positive there. In quadrants III and IV the sine is negative because the vertical coordinates are negative there.

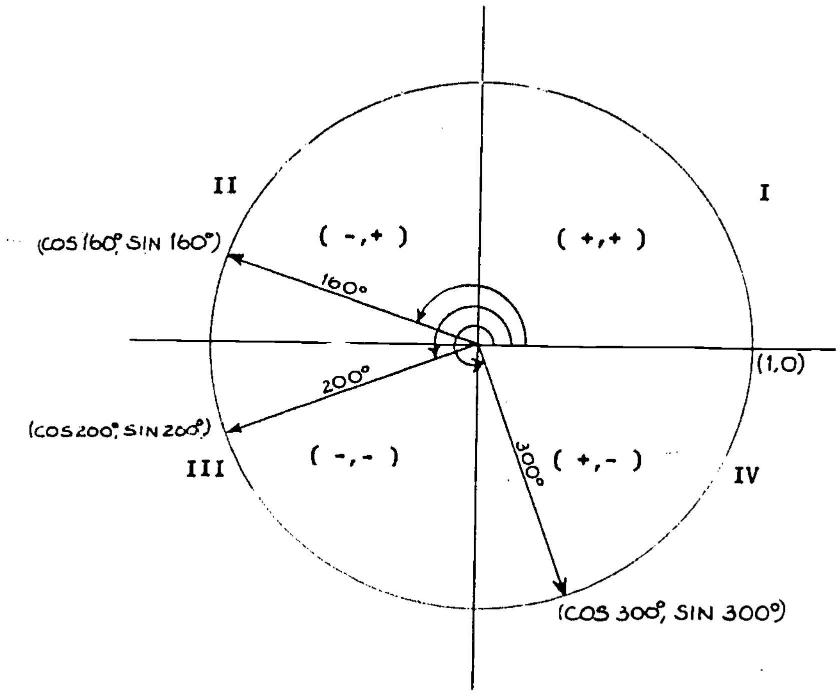
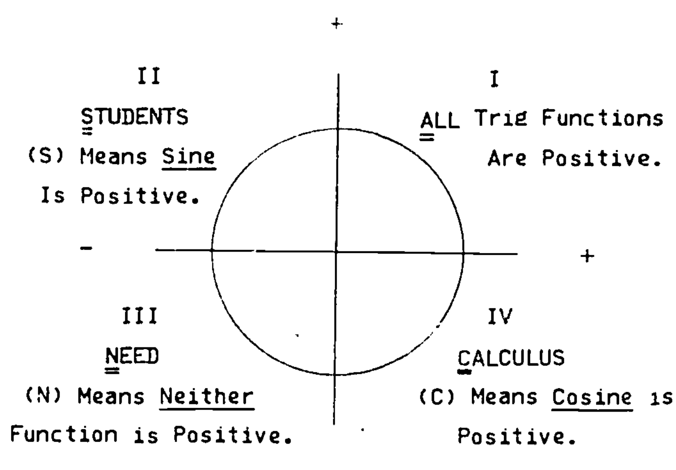


FIGURE 2

17-4 A Memory Trick to Remember the Signs of the Sine and Cosine

Cosines are positive for angles between 0° and 90° and between 270° and 360°. Angles between 90° and 270° have negative cosines. Sines are positive between 0° and 180° and negative between 180° and 360°. The signs of these functions may always be determined from the signs of the x- and y-components of the polar vectors, but there is also a memory device if you prefer memorizing to thinking. It is the statement "All Students Need Calculus." You may question the truth of the statement, but this is how it works. "All" represents the first quadrant, where all trigonometric functions are positive. The S which begins the word "Students" stands for the sine, which is positive in the second quadrant. Neither function is positive in the third quadrant,



ALL STUDENTS NEED CALCULUS

which we are told by the N with which "Need" begins. And the C of "Calculus" tells us that in the fourth quadrant, the cosine is positive. Once we know its secret, the statement "All Students Need Calculus" tells us which functions are positive in each of the four quadrants. All this is summarized in the figure on the preceding page.

PROBLEM SET 17:

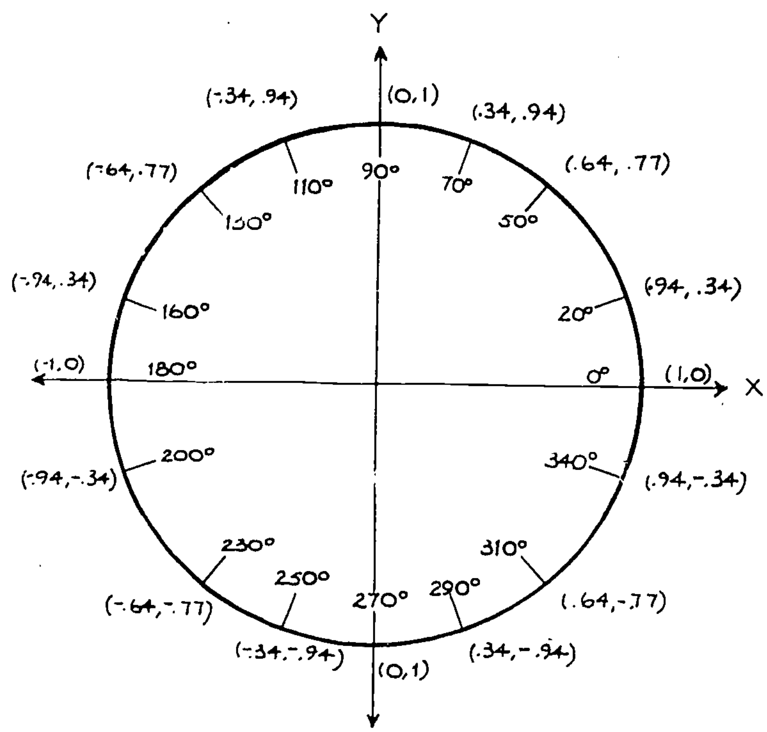


FIGURE 1

Refer to Figure 1 above for Problems 1 through 15.

1. What is the value of cos 20°?
2. sin 20° ≈
3. sin 70° ≈
4. cos 230° ≈
5. sin 270° ≈
6. cos 340° ≈
7. sin 90° ≈
8. cos 0° ≈

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9. The cosine of an angle is given by the ___-coordinate of the ordered pair.

10. The sine of an angle is given by the ___-coordinate of the ordered pair.

11. Complete the table.

Quadrant II	Quadrant I
sin is (?)	sin is (+)
cos is (?)	cos is (+)
sin is (?)	sin is (?)
cos is (?)	cos is (?)
Quadrant III	Quadrant IV

12. a. The lar[illegible] value for cos θ is ______.

    b. At which angle does this value occur: 0°, 90° or 250°?

13. a. The smallest value for cos θ is _____.

    b. This value occurs at (0°, 90° or 180°).

14. a. The largest value for sin θ is _____.

    b. This value occurs at (0°, 90° or 270°).

15. a. The smallest value for sin θ is _____.

    b. This value occurs at (0°, 90° or 270°).

Refer to Figure 2 on the following page to answer the following questions. All angles are in radians.

16. cos .5 ≈	21. cos $\frac{3\pi}{2}$ ≈	26. sin $\frac{3\pi}{4}$ ≈	31. sin 2π ≈
17. sin 2.36 ≈	22. sin $\frac{5\pi}{4}$ ≈	27. cos $\frac{2\pi}{3}$ ≈	32. cos π ≈
18. cos 4 ≈	23. cos 5.24 ≈	28. sin $\frac{7\pi}{6}$ ≈	33. sin 1.5 ≈
19. sin $\frac{11\pi}{6}$ ≈	24. cos 5 ≈	29. sin $\frac{4\pi}{3}$ ≈	34. sin $\frac{\pi}{2}$ ≈
20. sin 6 ≈	25. sin 3.5 ≈	30. cos $\frac{7\pi}{4}$ ≈	35. sin $\frac{3\pi}{2}$ ≈

36. The cosine is negative for angles between ____ radians and ____ radians.

37. The sine is positive for angles between ____ radians and ____ radians.

38. Convert the following angles in radians to their decimal approximations.

a. $\frac{\pi}{4}$	c. $\frac{5\pi}{4}$	e. $\frac{5\pi}{6}$	g. $\frac{\pi}{3}$	i. π	k. $\frac{5\pi}{3}$
b. $\frac{3\pi}{4}$	d. $\frac{\pi}{6}$	f. $\frac{7\pi}{6}$	h. $\frac{2\pi}{3}$	j. $\frac{4\pi}{3}$	l. 2π

39. Convert the angles in Problem 38 to their equivalents in degree measure.

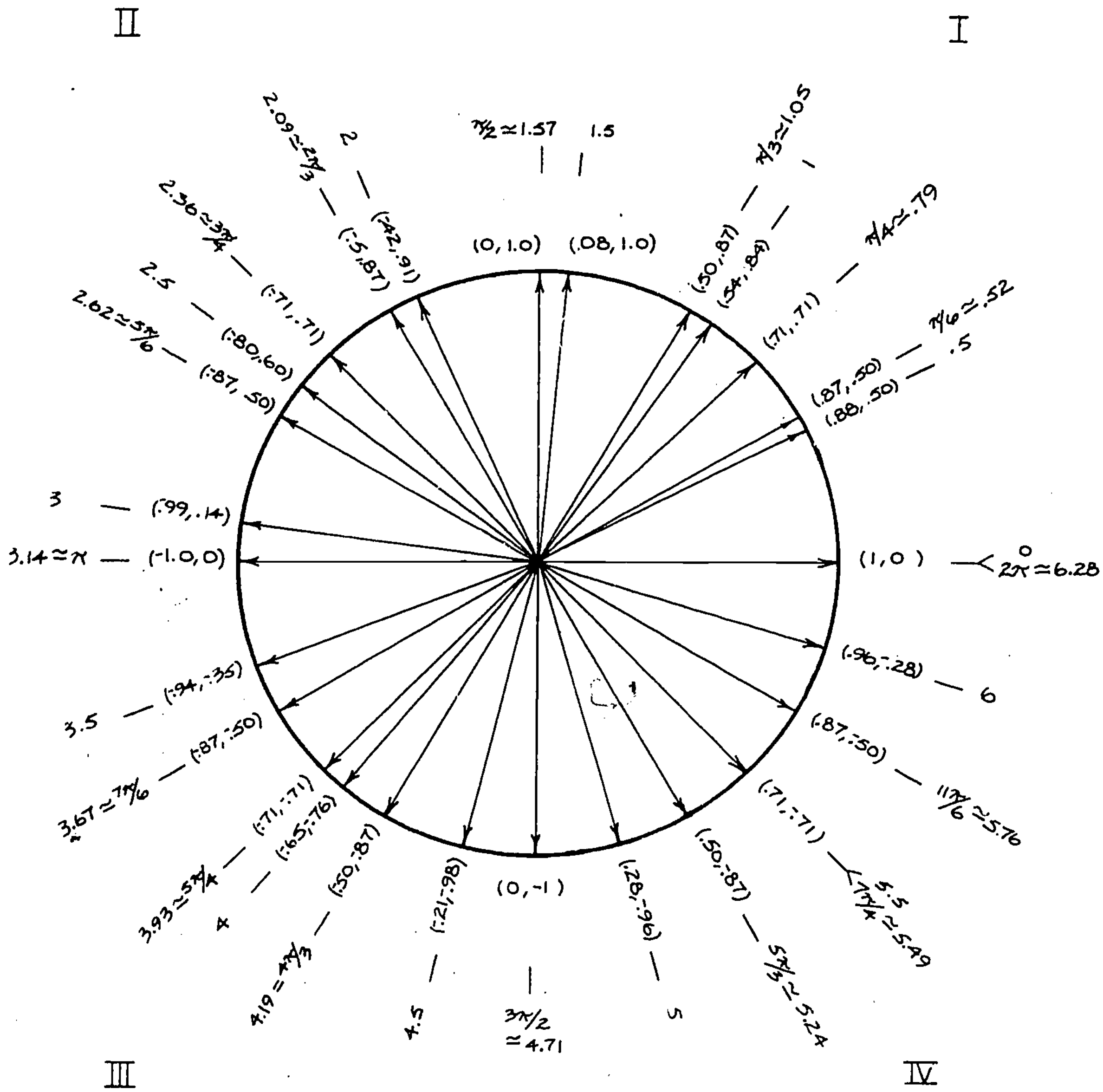


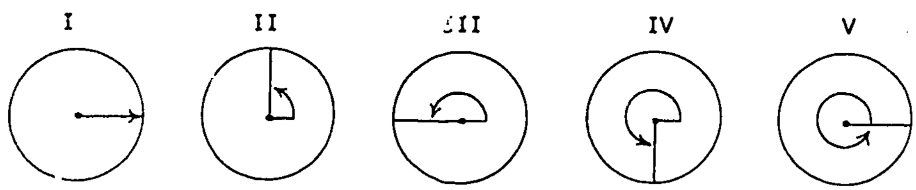
FIGURE 2

All angles are in units of radians.

SECTION 18: A GRAPH OF $y = \sin x$

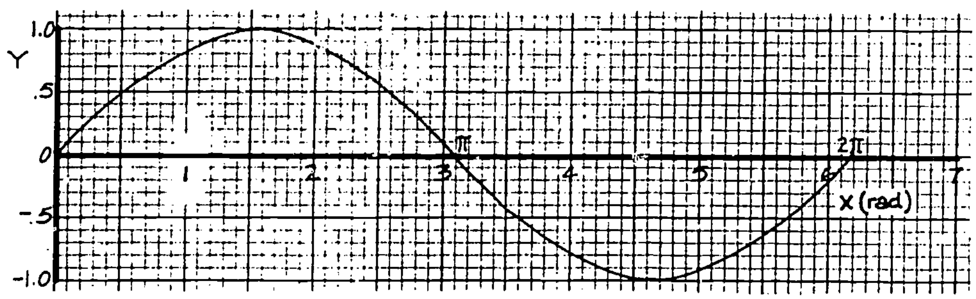
18-1 A One-Handed Backward-Running Clock

Imagine a unit vector rotating around the origin like the hand of a clock running backwards. In order to graph $y = \sin x$, we will keep track of two variables. One will be the angle through which the vector has rotated. This angle we will call the x-variable. The sine of each value of x we will call the y-variable. Below we have the first five snapshots of our rotating vector.



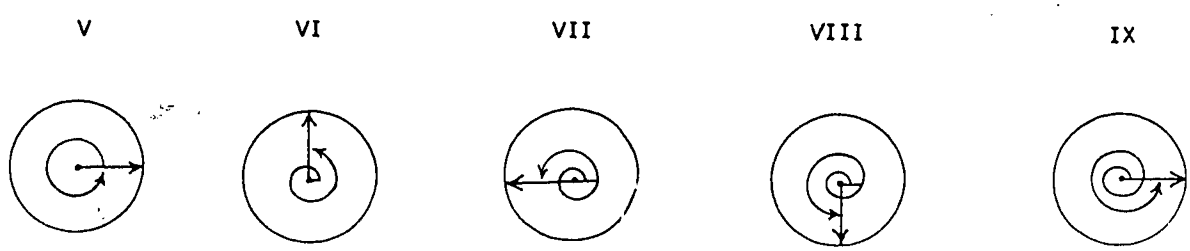
I	II	III	IV	V
$x = 0$ rad	$x = \frac{\pi}{2}$ rad	$x = \pi$ rad	$x = \frac{3\pi}{2}$ rad	$x = 2\pi$ rad
$y = 0$	$y = 1$	$y = 0$	$y = -1$	$y = 0$

The ordered pairs thus produced (along with others) may be graphed to produce a curve with the following shape. The curve is called a "sine wave."



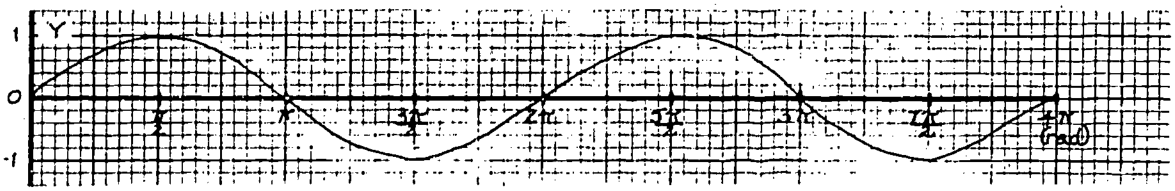
The first problem in the Problem Set requests you to graph just such a sine wave.

Notice that the x-axis does not stop at 2π. Therefore it is reasonable to ask, "What is the appearance of the graph of $y = \sin x$ to the right of $x = 2\pi$ rad?" To answer this question we produce four more snapshots of the rotating vector.

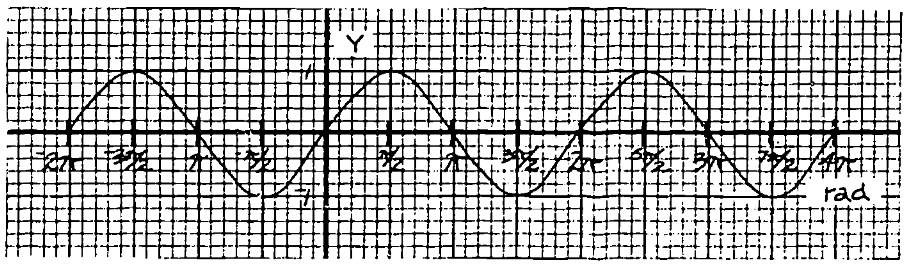


V	VI	VII	VIII	IX
$x = 2\pi$ rad	$x = (2\pi + \frac{\pi}{2})$ rad	$x = (2\pi + \pi)$ rad	$x = (2\pi + \frac{3\pi}{2})$ rad	$x = (2\pi + 2\pi)$ rad
$y = 0$	$y = 1$	$y = 0$	$y = -1$	$y = 0$

The ordered pairs produced may be graphed to show another segment of the graph of $y = \sin x$.



This result suggests that we could extend the graph of sin x for all real x's by merely linking together sine waves indefinitely in either direction. And indeed this is exactly what may be done.



18-2 Periodic Functions

Sin x is called a periodic function. It is a periodic function because the y-values have a pattern which repeats itself every time the x-values increase by 2π. We may state the periodic nature of sin x in a formula.

$$\sin x = \sin [x + n(2\pi)]$$

$$\text{for } n = 0, 1, -1, 2, -2, \ldots$$

To acquire a little insight into the meaning of this formula, we let

$$x = \frac{\pi}{2} \text{ rad}$$

and restrict ourselves to positive angles. Then

$$\begin{aligned} \sin \frac{\pi}{2} \text{ rad} &= \sin [\tfrac{\pi}{2} + (1 \cdot 2\pi)] \text{ rad} \\ &= \sin [\tfrac{\pi}{2} + (2 \cdot 2\pi)] \text{ rad} \\ &= \sin [\tfrac{\pi}{2} + (3 \cdot 2\pi)] \text{ rad} \\ &= \sin [\tfrac{\pi}{2} + (4 \cdot 2\pi)] \text{ rad} \end{aligned}$$

In other words, the formula says that we may add any number of complete revolutions to the angle without changing the value of the function. The reverse of the previous statement is also true. That is, any number of complete revolutions may be subtracted from an angle without changing the value of the sine. This idea is useful for computing the sine of angles larger than 2π. Some examples follow.

EXAMPLE:

Find sin 17π rad.

SOLUTION:

First we subtract from 17π all the complete revolutions. Since each revolution is 2π rad, even numbers of π's will be complete revolutions.

$$
\begin{aligned}
17\pi \text{ rad} &= 16\pi \text{ rad} + \pi \text{ rad} \\
&= 8 \text{ revolutions} + \pi \text{ rad}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sin 17\pi \text{ rad} &= \sin (\pi + 16\pi) \text{ rad} \\
&= \sin [\pi + (8 \cdot 2\pi)] \text{ rad} \\
&= \sin \pi \text{ rad} \\
&= 0
\end{aligned}
$$

EXAMPLE:

Find sin 17.56 rad.

SOLUTION:

First we determine the number of complete revolutions. A revolution is about 6.28 rads. We may subtract multiples of 6.28 from 17.56 to find the number of complete revolutions in 17.56 rad.

$$
\begin{aligned}
1 \cdot 6.28 &= 6.28 \\
2 \cdot 6.28 &= 12.56 \\
& \qquad \longleftarrow 17.56 \\
3 \cdot 6.28 &= 18.84
\end{aligned}
$$

This information tells us that an angle of 17.56 rad is between 2 and 3 complete revolutions. To find the fraction of a revolution left we subtract 12.56 from 17.56.

$$
\begin{array}{r}
17.56 \text{ rad} \\
\underline{-\ 12.56 \text{ rad}} \\
5.00 \text{ rad}
\end{array}
$$

In other words,

$$
\begin{aligned}
17.56 \text{ rad} &= 12.56 \text{ rad} + 5 \text{ rad} \\
& \quad 2 \text{ revolutions} + 5 \text{ rad}
\end{aligned}
$$

Consequently,

$$\sin 17.56 \text{ rad} \approx \sin 5 \text{ rad}$$

We may find sin 5 rad by referring to Figure 2 in Problem Set 8.

$$\sin 5 \text{ rad} \approx -.96$$

Therefore,

$$\sin 17.56 \text{ rad} \approx -.96$$

EXAMPLE:

Find sin 750°.

SOLUTION:

One revolution is 360°; two are 720°.

$$750° = 720° + 30°$$
$$= 2 \text{ rev} + 30°$$

Therefore,

$$\sin 750° = \sin 30°$$
$$\sin 30° = .5$$

and, consequently,

$$\sin 750° = .5$$

PROBLEM SET 18:

1. a. Copy and complete the table. Refer to Figure 2 on page 90 for values of sin x.

x (rad)	x (degrees)	y = sin x
0	0	0
.52	30	
.76	45	
1.00	57	.84
1.05	60	
1.50	86	1.00
1.57	90	
2.00	115	.91
2.09	120	
2.36	135	
2.50	143	.60
2.62	150	
3.00	172	.14
3.14	180	

x (rad)	x (degrees)	y = sin x
3.50	201	-.35
3.67	210	
3.93	225	
4.00	229	-.76
4.19	240	
4.50	258	-.98
4.71	270	
5.00	286	-.96
5.24	300	
5.50	315	
5.76	330	
6.00	344	-.28
6.28	360	

b. Construct a graph of y = sin x for the domain $0 \le x \le 6.28$ rad. Use these scales for your axes.

x-axis: 1 rad = 4 cm
y-axis: 1 unit = 4 cm

Note: Depending on your graph paper, it will probably be necessary to tape two sheets of graph paper together.

2. Construct a graph of $y = \sin x$ for the domain $0 \leq x \leq 12.56$ rad. Use these scales for your axes.

x-axis: 1 rad = 2 cm

y-axis: 1 unit = 2 cm

Note: Once again, it may be necessary to extend your graph by taping on a second sheet.

3. $\sin 37° = \sin (360° + 37°)$
$= \sin 397°$

$\sin 37° = \sin (720° + 37°)$
$= \sin (757°)$

Find <u>three more</u> angles that have the same sine as 37°.

4. $\sin (\frac{\pi}{4} \text{ rad}) = \sin [(2\pi + \frac{\pi}{4}) \text{ rad}]$
$= \sin (2\frac{1}{4}\pi \text{ rad})$

$\sin (\frac{\pi}{4} \text{ rad}) = \sin [(4\pi + \frac{\pi}{4}) \text{ rad}]$
$= \sin (4\frac{1}{4}\pi \text{ rad})$

Find <u>three more</u> angles that have the same sine as $\frac{\pi}{4}$ rad.

5. $\sin (2.5 \text{ rad}) \approx \sin [(6.28 + 2.5) \text{ rad}]$
$\approx \sin (8.78 \text{ rad})$

$\sin (2.5 \text{ rad}) \approx \sin [(12.56 + 25) \text{ rad}]$
$\approx \sin (15.06 \text{ rad})$

Find <u>three more</u> angles that have the same sine as 2.5 rad.

6. a. $4\frac{1}{6}\pi$ rad = u complete revolutions + v rad

Determine the values of u and v in the above equation.

b. $\sin (4\frac{1}{6}\pi \text{ rad}) = \sin (v \text{ rad})$

$\sin (4\frac{1}{6}\pi \text{ rad}) = ?$ (Answer is a number)

To answer Problems 7 through 20 it will often be helpful to refer to the completed table of Problem 1a.

7. a. 22.35 rad ≈ u revolutions + v rad

Determine the values of u and v in the above equation.

b. $\sin (22.35 \text{ rad}) = \sin (v \text{ rad})$

$\sin (22.35 \text{ rad}) = ?$ (Answer is a number)

8. a. 921° = u revolutions + v°

Determine the values of u and v in the above equation.

b. $\sin 921° = \sin v°$

$\sin 921° = ?$ (Answer is a number)

9. $\sin 540° =$	13. $\sin (20.84 \text{ rad}) \approx$	17. $\sin 3630° =$
10. $\sin (7.78 \text{ rad}) \approx$	14. $\sin 1857° \approx$	18. $\sin 360{,}057° \approx$
11. $\sin (17\frac{3}{4}\pi \text{ rad}) \approx$	15. $\sin (46\frac{1}{6}\pi \text{ rad}) \approx$	19. $\sin (19\frac{1}{3}\pi \text{ rad}) \approx$
12. $\sin 800° \approx$	16. $\sin (23.84 \text{ rad}) \approx$	20. $\sin (1001\frac{2}{6}\pi \text{ rad}) \approx$

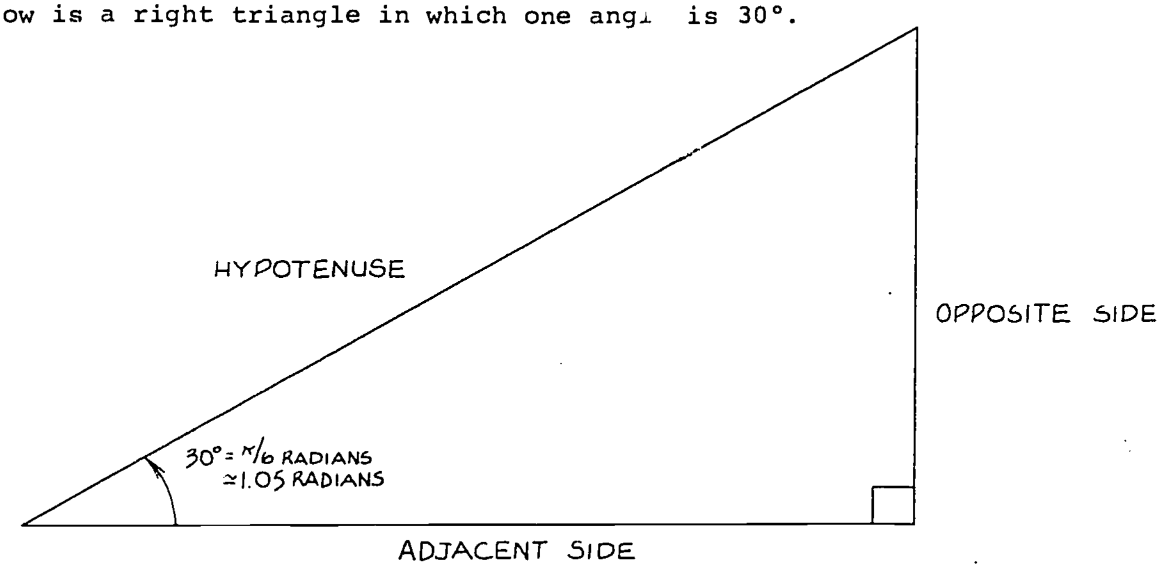
SECTION 19: TRIGONOMETRIC FUNCTIONS AND RIGHT TRIANGLES

19-1 Cosines, Sines and Right Triangles

You were introduced to the concept of similar figures in Section 11, and in Section 12 we restricted ourselves to similar triangles. We stated three theorems that may be used to determine whether pairs of triangles are similar. If two triangles are similar, unknown sides or angles may be found by knowing corresponding sides and angles of a similar triangle.

In the previous sections we introduced the trigonometric functions cosine and sine. These functions are extremely useful in studying triangles. In this section we will show how cosines and sines of angles express the relations between various parts of a right triangle.

Below is a right triangle in which one angle is 30°.



The side opposite the right angle is the hypotenuse. We call the other side adjacent (next) to the 30° angle the "adjacent side" and the side opposite the 30° angle the "opposite side." Let us measure the lengths of the three sides.

The length of the adjacent side is 10.4 cm, the length of the opposite side is 6 cm, while the hypotenuse is 12 cm long.

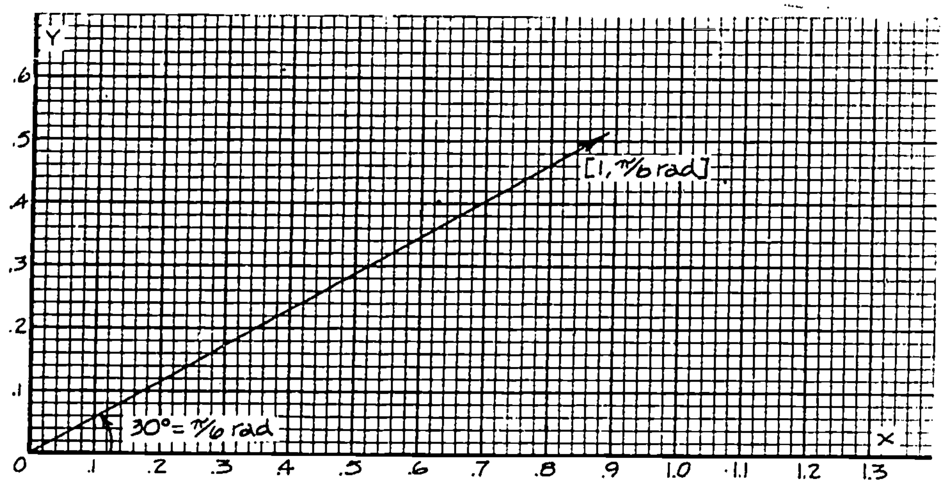
First we examine the ratios of the other sides to the hypotenuse. The ratio of the adjacent side to the hypotenuse is

$$\frac{\text{adjacent side}}{\text{hypotenuse}} \simeq \frac{10.4 \text{ cm}}{12 \text{ cm}} \simeq .87$$

The ratio of the opposite side to the hypotenuse is

$$\frac{\text{opposite side}}{\text{hypotenuse}} \simeq \frac{6 \text{ cm}}{12 \text{ cm}} = .50$$

Recall how we determined the cosine and sine of $\frac{\pi}{6}$ radians (= 30°). We graphed the polar vector $[1, \frac{\pi}{6} \text{ rad}]$ and measured its x- and y-components. This vector is shown on the following graph.



The x-component of the vector is approximately .87, and the y-component is approximately .50. Therefore,

$$\cos 30° \simeq .87$$
$$\sin 30° \simeq .50$$

Compare the cosine of 30° to the ratio of the adjacent side of our right triangle to the hypotenuse. Compare the sine of 30° to the ratio of the opposite side to the hypotenuse, The ratio of the adjacent aide to the hypotenuse is equal to the cosine of 30°, and the ratio of the opposide side to the hypotenuse is equal to the sine of 30°.

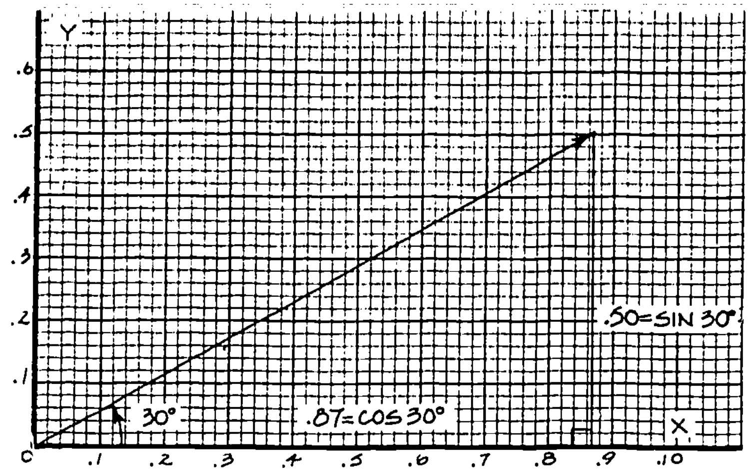
$$\frac{\text{adjacent side}}{\text{hypotenuse}} = \cos 30°$$

$$\frac{\text{opposite side}}{\text{hypotenuse}} = \sin 30°$$

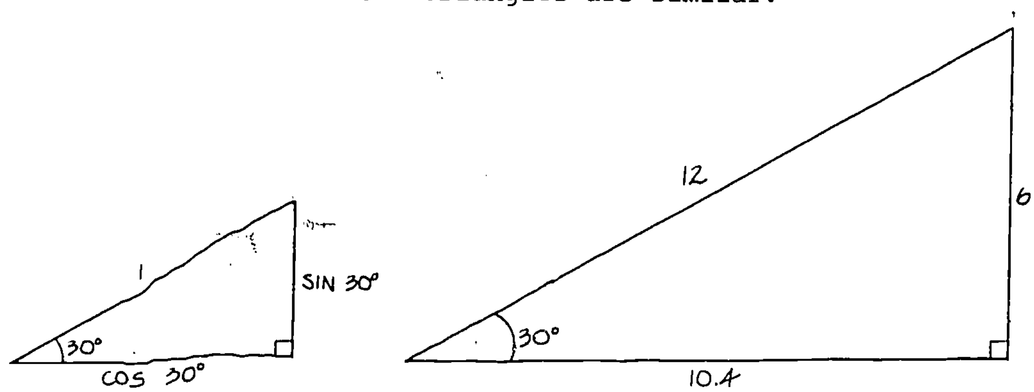
Is this a coincidence? Is it true for all right triangles?

We can show that these equalities are not coincidence and that they are true for all right triangles. We draw a vertical line (see graph, following page) from the head of the vector [1, 30°] to the x-axis to make a triangle. (Recall that the length of the vector is one unit on either the x- or y-axis.

This vertical line is the side of the triangle opposite the 30° angle. Its length is equal to the y-component of the vector, which is the sine of 30°. The side adjacent to the 30° angle is the x-component of the vector; it is the cosine of 30°.



If two angles of a triangle are equal to two angles of another triangle, the triangles are similar. The triangle that we just constructed contains a 30° angle and a right angle. So does the right triangle with which we begin this discussion, the triangle with sides of 6, 10.4 and 12 cm. Therefore, these two triangles are similar.



If two triangles are similar, their sides are proportional. The ratio of the adjacent side of triangle B to its hypotenuse is equal to the ratio of the adjacent side of triangle A to its hypotenuse.

$$\frac{\text{adjacent side of B}}{\text{hypotenuse of B}} = \frac{\text{adjacent side of A}}{\text{hypotenuse of A}}$$

$$\frac{10.4\text{ cm}}{12\text{ cm}} \approx \frac{\cos 30°}{1} = \cos 30°$$

Similarly, the ratio of the opposite side of B to its hypotenuse is equal to the ratio of the opposite side of A to its hypotenuse.

$$\frac{\text{opposite side of B}}{\text{hypotenuse of B}} = \frac{\text{opposite side of A}}{\text{hypotenuse of A}}$$

$$\frac{6\text{ cm}}{12\text{ cm}} \approx \frac{\sin 30°}{1} = \sin 30°$$

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Thus the ratio of the adjacent side to the hypotenuse of a 30° right triangle is equal to the cosine of 30°, and the ratio of the opposite side to the hypotenuse is equal to the sine of 30°. This is so because of the properties of similar triangles.

We can show that these equalities are not restricted to 30° right triangles by considering a right triangle with any angle θ (the Greek letter "theta"). By substituting the letter θ for 30°, the above argument becomes completely general. In other words, it will work for any angle.

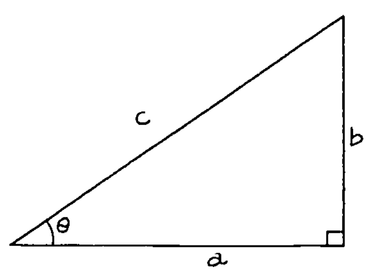
In summary, for any right triangle with an angle θ, the ratio of the adjacent side to the hypotenuse is equal to the cosine of θ, and the ratio of the opposite side to the hypotenuse is equal to the sine of θ.

If you have difficulty remembering that the cosine is associated with the adjacent side while the sine is associated with the opposite side, you may wish to note that "adjacent" and "cosine" occur alphabetically before "opposite" and "sine" do. And after introducing a third trigonometric function, we will mention another memory device. In the next session we will also give a few examples illustrating the relations between trigonometric functions and the sides of right triangles.

19-2 <u>The Tangent Function</u>

The cosine of an angle is the ratio of the side adjacent to the angle to the hypotenuse, and the sine of an angle is the ratio of the side opposite the angle to the hypotenuse. However, we have not yet introduced a function that relates the adjacent side and the opposite side of a right triangle.

This function is called the <u>tangent</u>. The tangent of an angle in a right triangle is the ratio of the <u>opposite</u> side to the <u>adjacent</u> side.

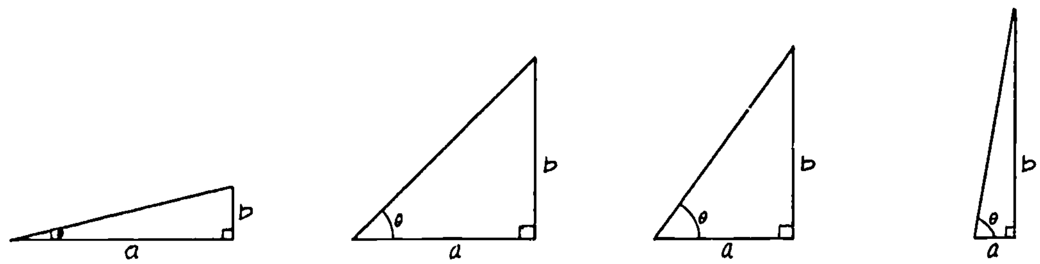


$$\text{tangent of } \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

The tangent of angle θ in the triangle above is the ratio of b to a. Tangent is abbreviated "tan," so we may write

$$\tan\theta = \frac{b}{a}$$

The tangent of 0° is 0, since the opposite side b is equal to 0. As θ increases, the tangent also increases. This may be seen by considering the series of right triangles illustrated on the following page. Observe that as θ increases, the ratio of b to a also increases.

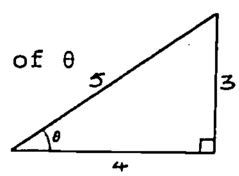


As θ approaches 90°, the tangent of θ becomes very large, because b becomes much greater than a. As θ becomes closer to 90°, side "a" becomes closer to zero. Since tangent $\theta = \frac{b}{a}$, the tangent of 90° is not defined because division by zero is not defined. We will not concern ourselves with tangents of angles greater than 90°.

We will illustrate how the cosine, sine and tangent can be determined from the sides of right triangles by a few specific examples.

EXAMPLE:

Find the cosine, sine and tangent of θ in the following right triangle.

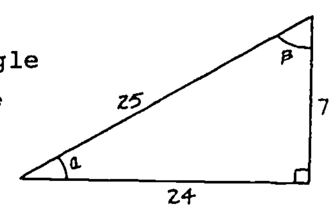


SOLUTION:

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{4}{5} = .8 \qquad \sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{3}{5} = .6 \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{3}{4} = .75$$

EXAMPLE:

Find the cosine, sine and tangent of both angle α and angle β in the triangle to the right. (α is the Greek letter alpha; it corresponds to our letter A. β is the Greek letter beta and corresponds to our letter B. Our word "alphabet" comes from these two Greek letters.)



SOLUTION:

The side adjacent to α is 24; the side opposite is 7.

Therefore,

$$\cos\alpha = \frac{24}{25} \qquad \sin\alpha = \frac{7}{25} \qquad \tan\alpha = \frac{7}{24}$$

However, the side adjacent to β is not 24 but 7. The side opposite β is 24. Consequently,

$$\cos\beta = \frac{7}{25} \qquad \sin\beta = \frac{24}{25} \qquad \tan\beta = \frac{24}{7}$$

Again there is a memory problem; remembering that the tangent of an angle is the opposide side divided by the adjacent side. Our memory device is another inane sentence: "Alice Has Opened Her Own Account." Associate the first letter of each word with the first letter of "adjacent," "opposite" or "hypotenuse."

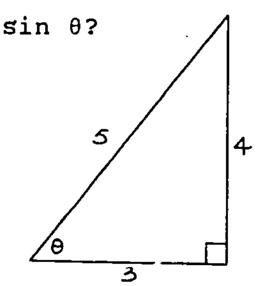
Alice Has	$\dfrac{\text{adjacent}}{\text{hypotenuse}} = \cos\theta$	cosine, sine and tangent are in alphabetical order (c,s,t)
Opened Her	$\dfrac{\text{opposite}}{\text{hypotenuse}} = \sin\theta$	
Own Account	$\dfrac{\text{opposite}}{\text{adjacent}} = \tan\theta$	

Everything that we have said about the ratios of the different sides and the trigonometric functions applies equally well to either of the acute (<90°) angles in a right triangle. However, <u>none of it applies to the right angle itself</u>.

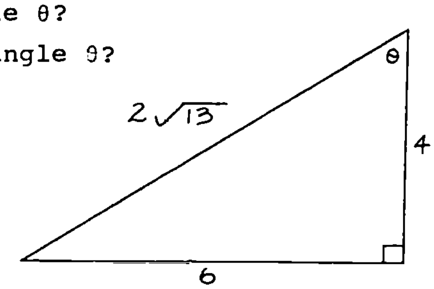
PROBLEM SET 19:

Write the missing words on your answer sheet.

1. Sin θ is given by the ratio of the (?) side to the hypotenuse.
2. Cos θ is given by the ratio of the (?) side to the hypotenuse.
3. Tan θ is given by the ratio of the (?) side to the (?) side.
4. a. In the triangle at the right, what is the value of sin θ?
   b. What is cos θ?
   c. What is tan θ?



5. a. In the triangle at the right, what is the length of the hypotenuse?
   b. What is the length of the side opposite angle θ?
   c. What is the length of the side adjacent to angle θ?
   d. sin θ = ? (rationalize the denominator)
   e. cos θ = ? (rationalize the denominator)
   f. tan θ = ?



6. a. $\sin \alpha =$

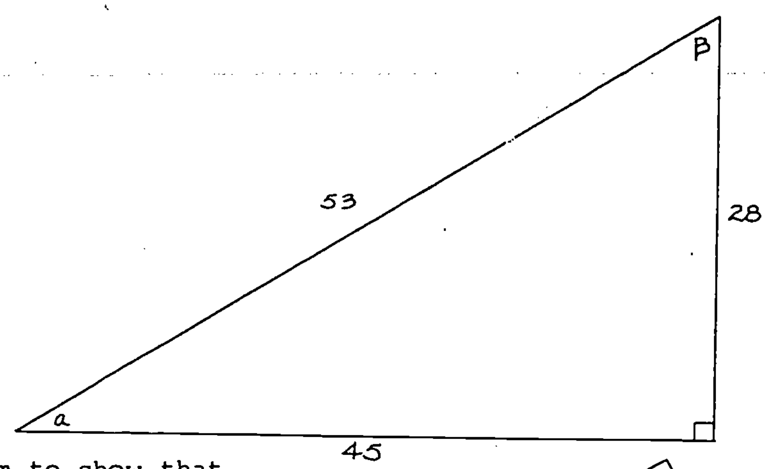
b. $\cos \alpha =$

c. $\tan \alpha =$

d. $\sin \beta =$

e. $\cos \beta =$

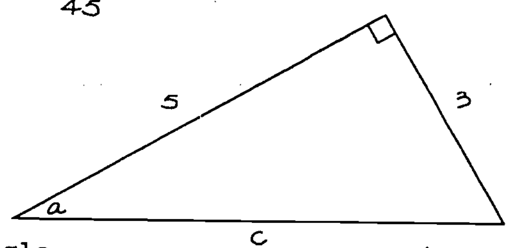
f. $\tan \beta =$



7. a. Use the Pythagorean Theorem to show that $c = \sqrt{34}$.

b. Therefore the value of $\sin \alpha$ is <u>(?)</u>. (Rationalize the denominator.)

c. $\cos \alpha =$ <u>(?)</u>. (Rationalize the denominator.)



8. The triangle at the right is a 45°-45°-90° triangle.

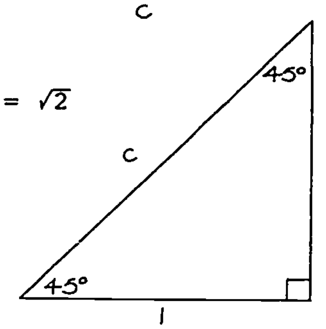
a. $\tan 45° =$

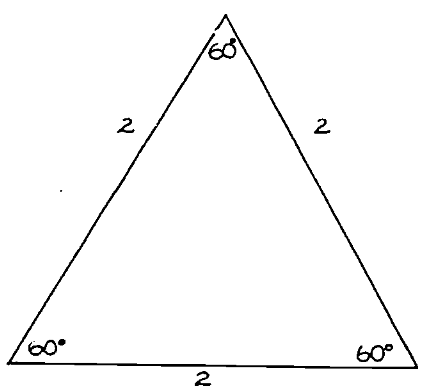
b. Show calculations leading to the conclusion that $c = \sqrt{2}$ (use the Pythagorean Theorem).

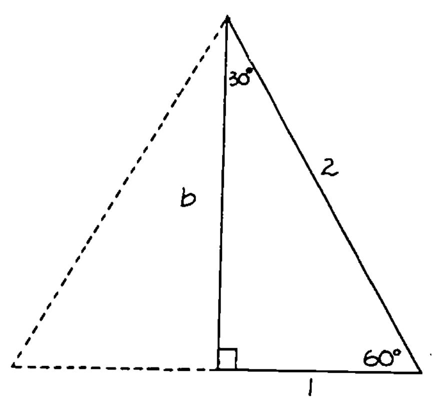
c. Therefore $\cos 45° =$

d. Show that when the denominator is rationalized the answer to Part c is $\frac{\sqrt{2}}{2}$.

e. Find $\sin 45°$ and rationalize the denominator.







9. The drawings above show how we can take an equilateral triangle and split it in half to make two 30°-60°-90° triangles. You can see that the 30°-60°-90° triangle has a hypotenuse of 2 and that one of the other sides has length 1.

a. We can see right away that $\sin 30° =$

b. Also $\cos 60° =$

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c. Show calculations leading to the conclusion that $b = \sqrt{3}$.

d. Now complete the table below. Be sure to rationalize denominators when necessary.

	30°	60°
sin θ	$\frac{1}{2}$	
cos θ		$\frac{1}{2}$
tan θ		

SECTION 20: THE USE OF TRIGONOMETRIC TABLES

20-1 Tables of Trigonometric Functions

We have defined three trigonometric functions in terms of the sides of a right triangle. For any angle θ of a right triangle,

	Memory Device
$\cos\ \theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$	Alice Has
$\sin\ \theta = \dfrac{\text{opposite side}}{\text{hypotenuse}}$	Opened Her
$\tan\ \theta = \dfrac{\text{opposite side}}{\text{adjacent side}}$	Own Account

You have learned how to determine approximate values of the cosine and sine of an angle by measuring the x- and y- components of a polar vector. The exact values of certain angles such as 30°, 45° and 60° may be determined geometrically. But trigonometric functions would be of limited use to us if only a few could be determined exactly and the rest could be found only by measuring.

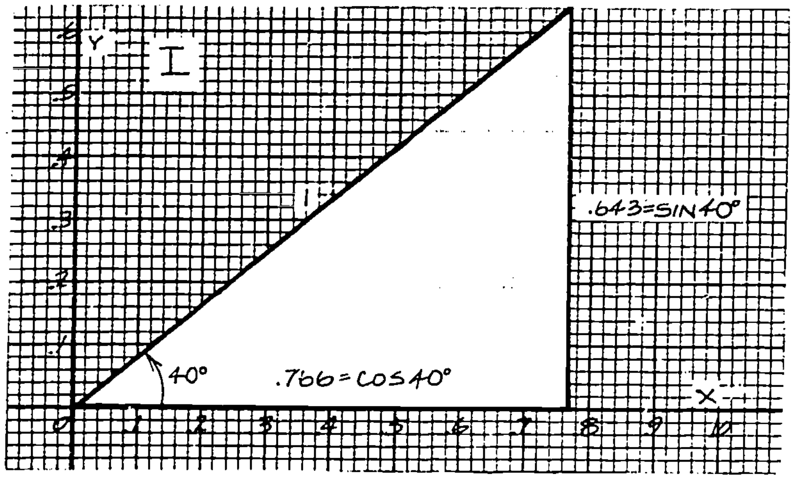
For these reasons, tables have been constructed that give values of sines, cosines and tangents exact to a certain number of decimal places. Advanced mathematical techniques are required to obtain these values, but once they are obtained, they are available for all of us to use. Such a table is given at the back of this book.

Angles are given in degrees, and the radian equivalent is also given. The values of trigonometric functions are given to three decimal places. However, with a few exceptions they are not exact, because they generally are irrational numbers--decimals that go on endlessly.

Observe that $\sin\ \theta$ increases from 0 to 1 as θ increases from 0° to 90°. Cos θ decreases from 1 to 0 as θ increases from 0 to 90°, and $\tan\ \theta$ increases from 0 without limit as θ increases to 90°.

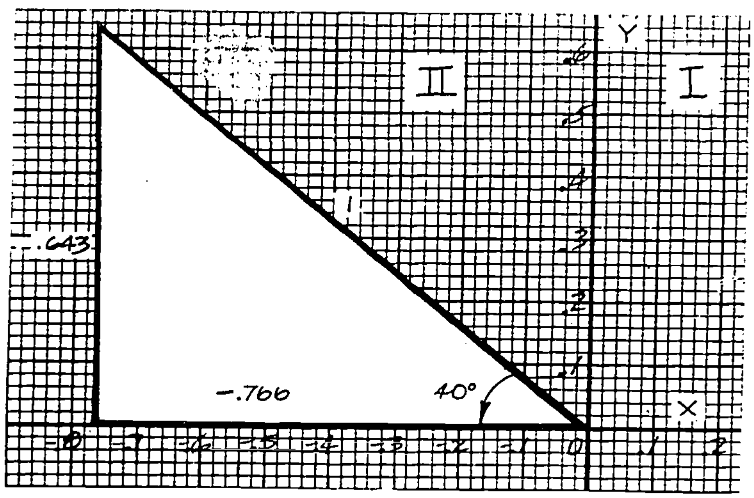
The fact about the table that is of most immediate interest to us is that the range of angles is only from 0° to 90°. However, the table may be used to determine trigonometric functions of all possible angles. We will now explain how to find the sines and cosines of angles greater than 90°.

On the graph on the following page is drawn a right triangle with an angle of 40° and a hypotenuse of 1. The side adjacent to the 40° angle is the cosine of 40°, and the side opposite the 40° angle is the sine of 40°. According to the Table of Trigonometric Functions the cosine of 40° is approximately .766 and the sine of 40° to three decimal places is .643.



20-2 The Second Quadrant

Suppose that we cut this triangle from the graph, turn it over and place it in the second quadrant as shown on the next graph.

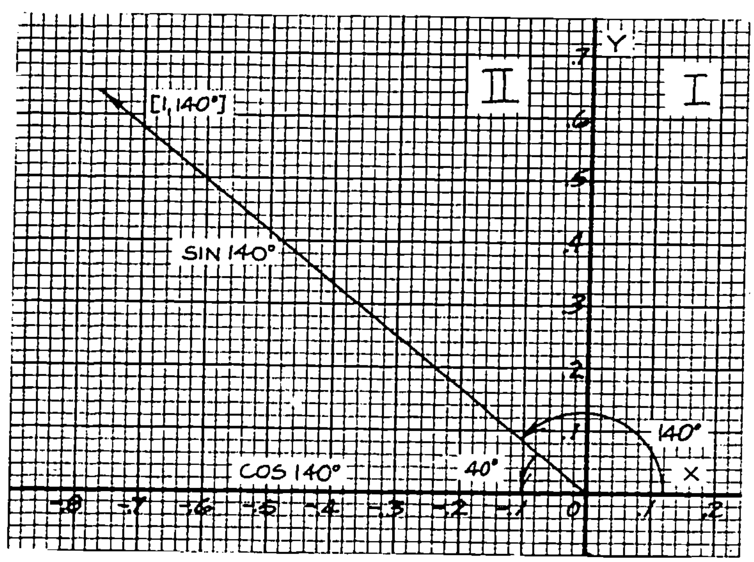


The dimensions of this triangle are the same as those of the original. The side adjacent to the 40° angle is .766 and the opposite side is .643.

Now let us graph the polar vector [1, 140°] (see graph at right).

The cosine of 140° is the x-component of this vector, and the sine of 140° is the y-component of this vector.

The sum of 140° and 40° is equal to 180°, and polar vector [1, 140°] coincides with the



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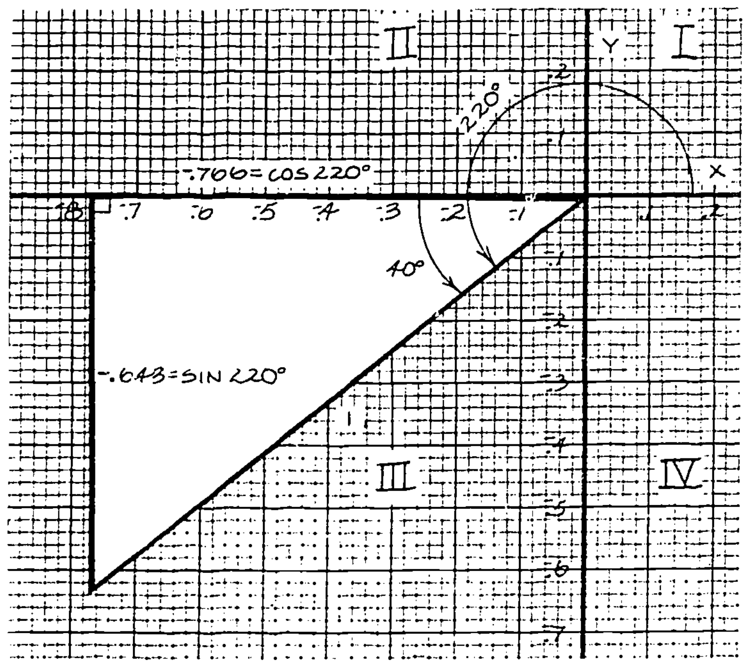
hypotenuse of the 40° right triangle placed in the second quadrant. The x-component of the polar vector is -1 times the length of the adjacent side of the triangle. Therefore, the cosine of 140° is -.766. The y-component of the polar vector is equal to the length of the side of the right triangle opposite the 40°. The sine of 140° is therefore approximately .643. The cosine and sine of 140°, except for + and - signs, are the same as the cosine and sine of 40°.

20-3 The Third Quadrant

We now place our cut-out 40° right triangle as shown on the graph below. Note that we always place the same side of the triangle along the x-axis. The length of the side adjacent to the 40° angle is .766; the length of the side opposite is .643.



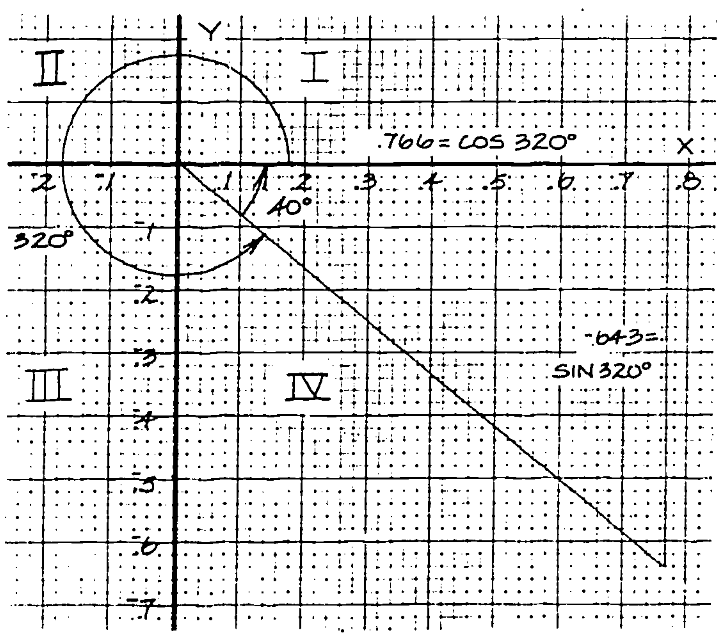
The polar vector [1, 220°] coincides with the hypotenuse of this triangle. Therefore, the x-component of this vector is -.766, and the y-component is -.643. These components are the cosine and sine of 220°, so

$$\cos 220° \approx -.766$$
$$\sin 220° \approx -.643$$

Observe that the cosine and sine of 220° are equal to the same functions of 40° except for the signs.

20-4 The Fourth Quadrant

We complete our trip around the quadrants by placing the 40° right triangle along the x-axis in the fourth quadrant of the graph to the right. The polar vector [1, 320°] coincides with the hypotenuse of this triangle. The x- and y- components of the vector are the cosine and sine of 320°, so



$$\cos 320° \approx .766$$
$$\sin 320° \approx -.643$$

Again, this cosine and sine are equal (except for sign) to the cosine and sine of 40°.

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20-5 Reference Angles

We have shown that it is possible to obtain values of the cosine and sine of 140°, 220° and 320° from the cosine and sine of 40°. We say that 40° is the reference angle for these other three angles. When we say that 40° is the reference angle, we mean that the other three angles are each 40° from the x-axis. 140° is 40° less than 180°; 220° is 40° greater than 180°; and 320° is 40° less than 360°.

The first step in finding the cosine or sine of an angle not in the first quadrant is to determine the reference angle. For angles in the second or third quadrants this is done by finding the difference between the angle and 180°. For angles in the fourth quadrant we find the difference between the angle and 360°.

The next step is to use the table to find the cosine or sine of the reference angle.

Finally, the cosine or sine is given the sign appropriate to the quadrant. The proper sign may be determined from that dreadful sentence, "All Students Need Calculus." All trigonometric functions are positive in the first quadrant; the sine is positive in the second quadrant; neither sine nor cosine is positive in the third quadrant; and the cosine is positive in the fourth quadrant.

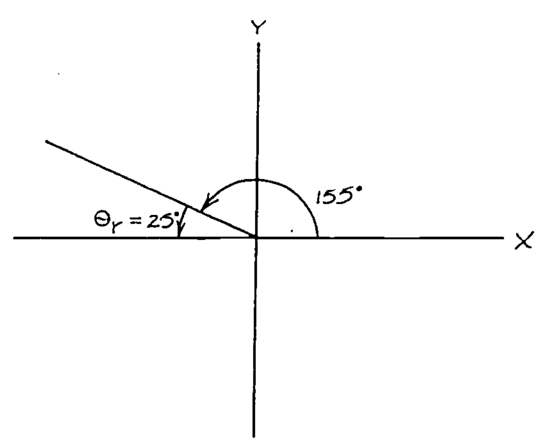
20-6 Examples

EXAMPLE:

Determine the cosine and sine of 155°

SOLUTION:

155° is in the second quadrant. The reference angle is 180° - 155° = 25°. We will often use the symbol θ_r for "reference angle."



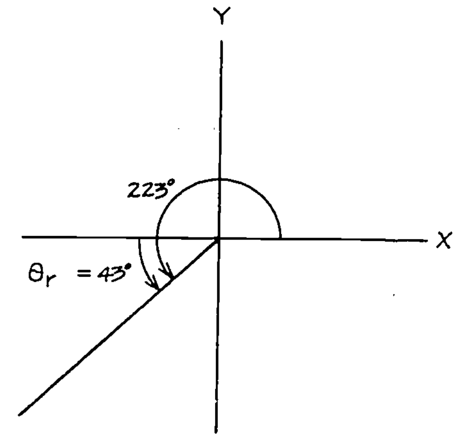
The cosine of 25° is .906. The sine of 25° is .423. Since 155° is in the second quadrant, the cosine is negative and the sine is positive. Thus cos 155° ≈ -.906 and sin 155° ≈ .423.

EXAMPLE:

Determine the cosine and sine of 223°.

SOLUTION:

223° is in the third quadrant. The reference angle is 223° - 180° = 43°. The cosine of 43° is .731, and the sine of 43° is .682. 223° is in the third quadrant, so both the cosine and sine are negative. Therefore, cos 223° ≈ -.731 and sin 223° ≈ -.682 (see graph to the right).

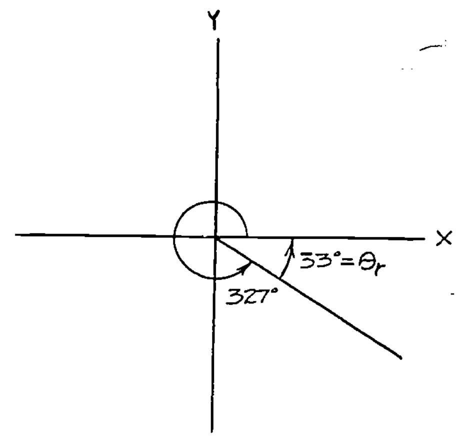


EXAMPLE:

Determine the cosine and sine of 327°.

SOLUTION:

Since 327° is in the fourth quadrant, the reference angle is 360° - 327° = 33° (see graph to the right). The cosine of 33° is .839; the sine is .545. In the fourth quadrant cosines are positive, while sines are negative. Consequently, cos 327° ≈ .839 and sin 327° ≈ -.545 (see graph to the right).

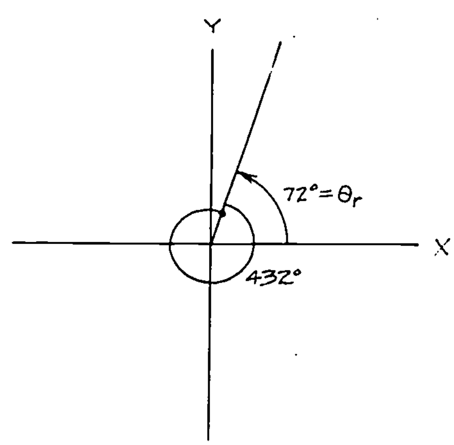


EXAMPLE:

Determine the cosine and sine of 432°.

SOLUTION:

The angle of 432° is in the first quadrant; it is equal to 360° + 72° (see graph at right). Polar vector [1, 432°] coincides with polar vector [1, 72°].



Therefore, 72° is the reference angle for 432°. Since 432° is in the first quadrant, its cosine and sine are positive.

$$\cos 432° = \cos 72° \approx .309$$

$$\sin 432° = \sin 72° \approx .951$$

<u>EXAMPLE</u>:

Determine the sine and cosine of 3.875 rad.

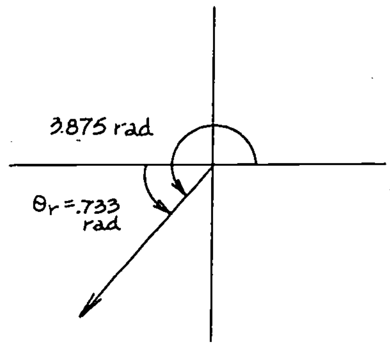
<u>SOLUTION</u>:

3.875 rad is in the third quadrant. The reference angle is 3.875 rad − π rad, or

$$3.875 \text{ rad} - 3.142 \text{ rad} = .733 \text{ rad}$$

$$\sin 3.875 \text{ rad} = -\sin .733 \text{ rad} \approx .669$$

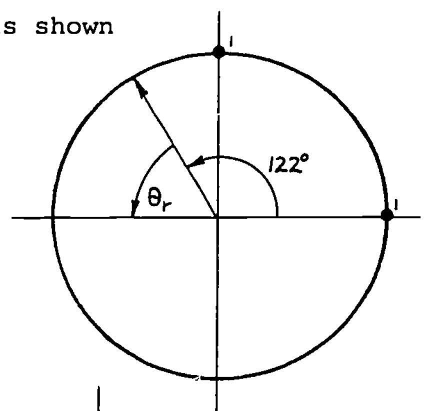
$$\cos 3.875 \text{ rad} = -\cos .733 \text{ rad} \approx .743$$



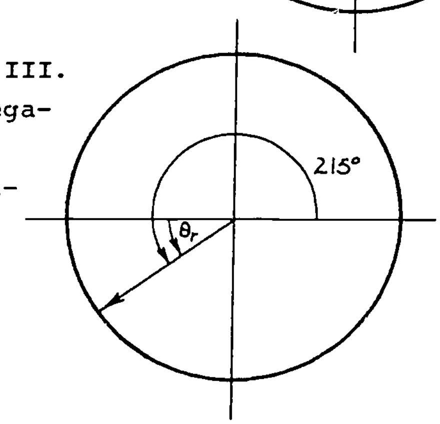
PROBLEM SET 20:

Use the TRIG TABLE at the end of the book to determine the values below.

1. $\sin 31° \approx$
2. $\cos 79° \approx$
3. $\tan 89° \approx$
4. $\cos 1.134$ rad $\approx$
5. $\tan .035$ rad $\approx$
6. sin. 663 rad $\approx$
7. a. The terminal vector of the 122° angle as shown to the right lies in quadrant <u>(?)</u> .
   b. $\theta_r = 58°$. Why?
   c. $\sin 58° \approx$
   d. $\cos 58° \approx$
   e. sin 122° will be (positive, negative).
   f. cos 122° will be (positive, negative).
   g. $\sin 122° \approx$
   h. $\cos 122° \approx$



8. a. $\theta_r =$
   b. The 215° angle lies in Quadrant III. Therefore sin 215° will be (positive, negative). (See graph at right.)
   c. cos 215° will be (positive, negative).
   d. $\sin 215° \approx$
   e. $\cos 215° \approx$

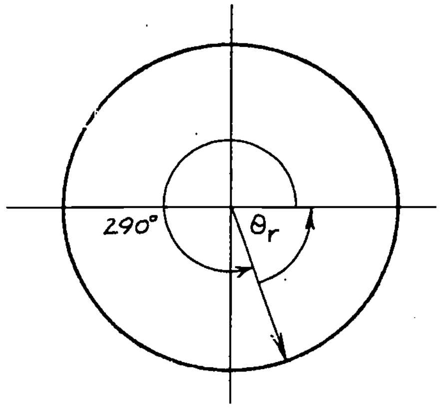


9. a. $\theta_r =$

b. $\cos 290° \approx$

c. $\sin 290° \approx$

(see graph at right)



Find the values of the following trigonometric functions. Use $\pi \approx 3.142$ and $2\pi \approx 6.283$.

10. cos 137°
11. sin 283°
12. cos 322°
13. sin 187°
14. cos 258°
15. cos 306°
16. sin 195°
17. sin 99°
18. cos 345°
19. sin 317°
20. sin 1.798 rad
21. cos 2.810 rad
22. sin 2.933 rad
23. cos 3.613 rad
24. sin 4.346 rad
25. cos 4.730 rad
26. sin 4.991 rad
27. sin 1.641 rad
28. sin 488°
29. cos 500°
30. sin 1000°

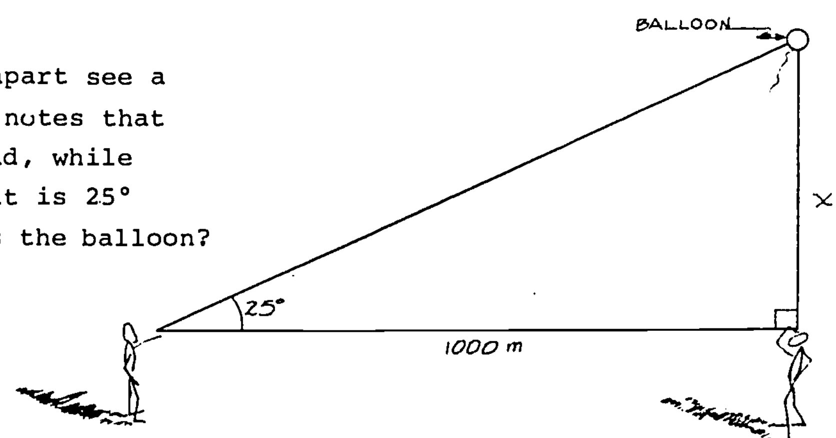
SECTION 21: USING TRIGONOMETRIC FUNCTIONS TO SOLVE PROBLEMS

In many situations certain sides and angles of right triangles are known, while other parts of the triangles are not known. Problems often occur in astronomy and navigation in which some angles and distances in a right triangle may be measured, but others cannot be. In these situations trigonometry can be used to find unknown dimensions.

A table of trigonometric functions is a catalog of the ratios of the sides of right triangles with various shapes. The following examples illustrate how a table of trigonometric functions can be used to solve for unknown parts of right triangles.

EXAMPLE:

Two children 1000 meters apart see a balloon in the sky. One child notes that the balloon is directly overhead, while the other child observes that it is 25° above the horizon. How high is the balloon? See graph at right.



SOLUTION:

The height of the balloon is the side of a right triangle; it is the side opposite an angle of 25°. Call this height x. The side adjacent to the 25° angle is 1000 meters. The tangent of an angle is the ratio of the opposite side to the adjacent side.

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

The tangent of 25° is the ratio of x to 1000 meters.

$$\tan 25° = \frac{x}{1000}$$

We consult the Trig Table at the end of the book and find that

$$\tan 25° \approx .466$$

Therefore,

$$\frac{x}{1000} \approx .466$$

We solve this equation for x.

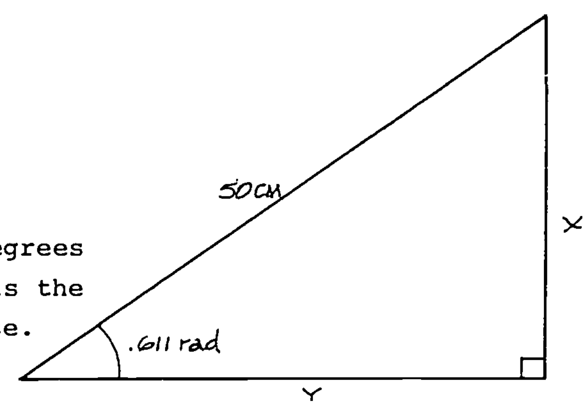
$$x \approx 1000 \cdot (.466)$$

$$x \approx 466 \text{ m}$$

The balloon is 466 meters above the ground.

EXAMPLE:

Solve for x and y.



SOLUTION:

Note that the angle is given not in degrees but in radians. The cosine of this angle is the ratio of the adjacent side to the hypotenuse.

$$\cos (.611 \text{ rad}) = \frac{y}{50}$$

The cosine of .611 rad is found from the table to be .819. Consequently,

$$\frac{y}{50} \approx .819$$

We solve this equation for y.

$$y \approx 50 \cdot (.819)$$

$$\approx 41 \text{ cm}$$

The sine of .611 rad is the ratio of the opposite side to the hypotenuse.

$$\sin (.611 \text{ rad}) = \frac{x}{50}$$

The sine of .611 rad is found from the table to be .574. Thus,

$$\frac{x}{50} \approx .574$$

$$x \approx 50 \cdot (.574)$$

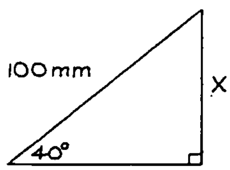
$$\approx 29 \text{ cm}$$

Note that having found either the opposite or adjacent sides by means of the trigonometric functions, you could find the remaining side by means of the Pythagorean Theorem.

PROBLEM SET 21:

In each case find the length of the unknown side x (with an uncertainty of ±.05, i.e., round to the nearest tenth).

1.



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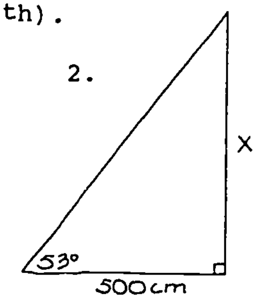

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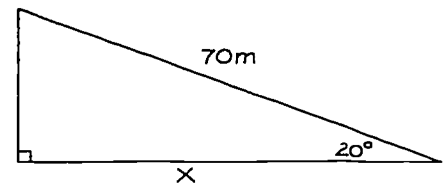

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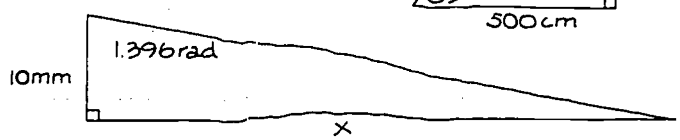

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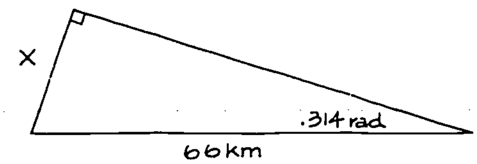
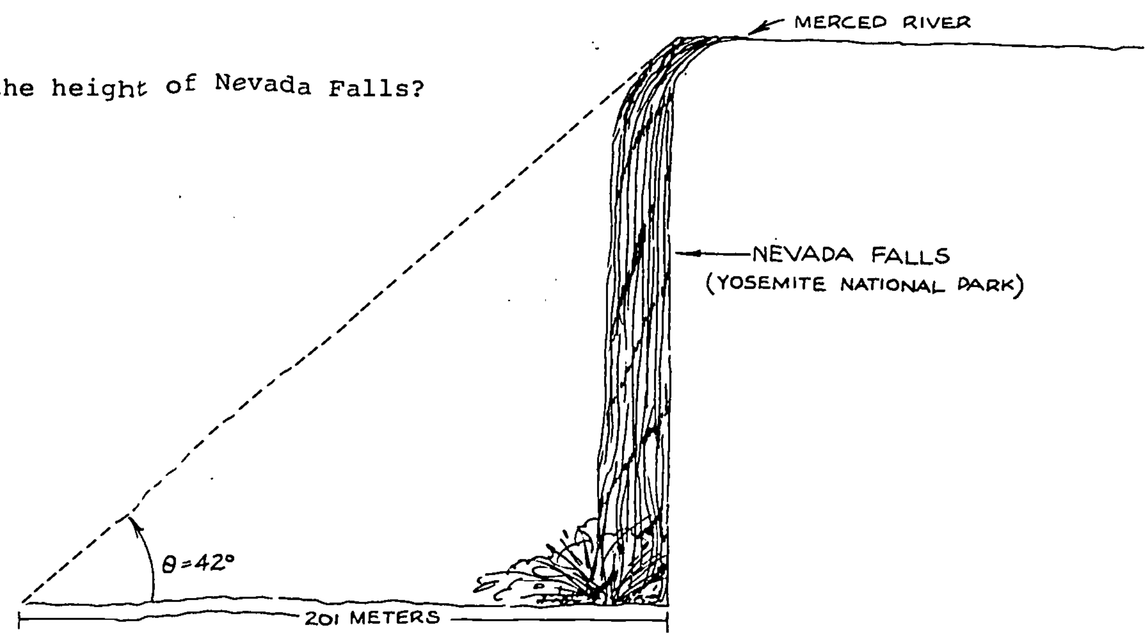
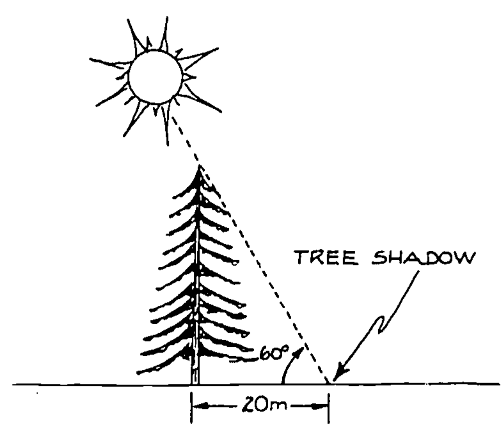


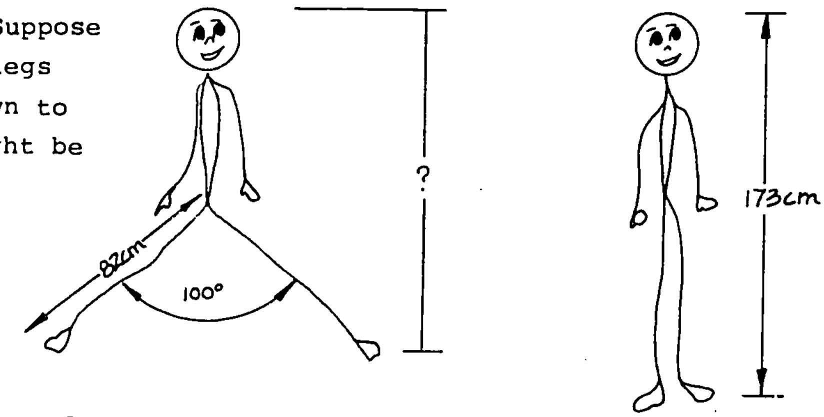
6. What is the height of Nevada Falls?



7. To find the height of a tree, a math student took the measurements indicated in the figure at right. Use these measurements to calculate the height of the tree.

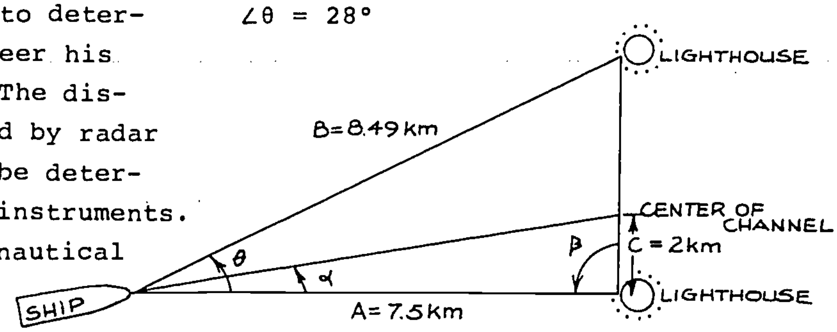


*8. Crocker Spaniel has a height of 173 cm and legs 82 cm long. Suppose that Crocker stands with his legs spread at a 100° angle as shown to the right. What will his height be in that position?



*9. A ship is approaching a channel to a harbor, as described in the illustration at right. The navigator needs to determine the angle α in order to steer his craft safely into the harbor. The distances A and B may be determined by radar measurements. The angle θ may be determined by visual sightings with instruments. The distance C is given on the nautical chart.

$A = 7.5$ km
$B = 8.49$ km
$C = 2$ km
$\angle\theta = 28°$



The navigator's problem is to determine $\angle\alpha$.

a. The navigator checks the ratio $\frac{A}{B}$ and finds that $\cos 28° \approx \frac{A}{B}$. This implies that $\angle\beta \approx 90°$. Why?

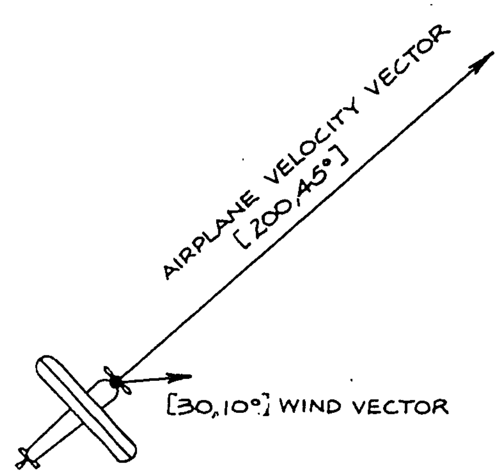
b. Calculate $\tan \alpha$ and then find the angle that corresponds to $\tan \alpha$ in the Trig Table.

*10. The true course and groundspeed of an airplane in flight may be determined by vector addition.

The pilot of an aircraft determines that he is heading due northeast with an airspeed of 200 km/hr. We may represent the magnitude and direction of his travel by means of the polar vector [200, 45°].

Next, the pilot radios the weather bureau and learns that the wind is blowing at a steady 30 km/hr in a direction 10° north of east. We may represent this information by the polar vector [30, 10°].

The vector sum of the two vectors will give the true direction and groundspeed of the aircraft. The diagram at right describes the two patterns of movement experienced by the airplane. The airplane is moving through the air. Simultaneously, the air itself is moving.



Graphical addition may be used to find the resultant, as was done in Section 7 for the addition of forces. However, now that you know how to use trigonometric tables, another method is available to you.

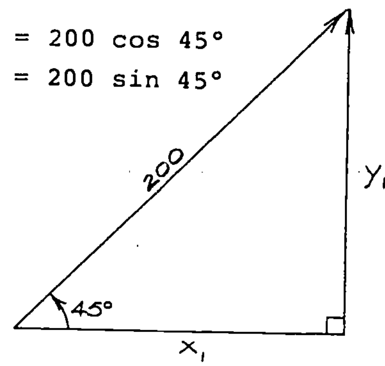
First, use the trigonometric functions cosine and sine to express each polar vector in its equivalent rectangular form.

Second, these rectangular vectors are added just as if they were nutritional vectors.

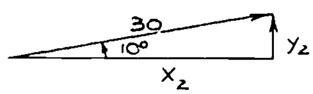
Third, the vector sum is converted back into polar form by means of the tangent, cosine and sine functions.

a. Convert the polar vector [200, 45°] into rectangular form (drawing at right).

$x_1 = 200 \cos 45°$
$y_1 = 200 \sin 45°$



b. Convert the polar vector [30, 10°] into its equivalent rectangular form (drawing below).



c. Add the vectors found in Parts a and b.

d. Determine tan θ for the vector sum.

e. In the tangent column of the Trig Table find the number closest to your answer to Part d. Then find the corresponding angle.

f. The magnitude of the sum may be found in a couple of ways, either by the Pythagorean Theorem, or by use of the angle found in Part e and the cosine or sine functions.

Find the magnitude of the sum.

REVIEW PROBLEM SET 22:

1. Plot the following polar vectors on the same graph.

a. [3, 20°]   c. [4.5, 175°]   e. [1.5, 500°]
b. [4, 75°]   d. [3.5, 410°]   f. [3, 630°]

2. Plot the following polar vectors on the same graph.

a. [4, $\frac{\pi}{6}$ rad]   c. [3, $\frac{5\pi}{6}$ rad]   e. [2.5, $\frac{7\pi}{3}$ rad]
b. [2, $\frac{\pi}{2}$ rad]   d. [5.5, $\frac{7\pi}{6}$ rad]   f. [1.5, $\frac{11\pi}{3}$ rad]

3. a. On graph paper, plot the point (5, 7).
b. Draw a vector from the origin to the point (5, 7).
c. Using the Pythagorean Theorem, determine the magnitude of the vector.
d. Determine the direction of the vector using a protractor.
e. The polar representation of the vector is [?, ?].

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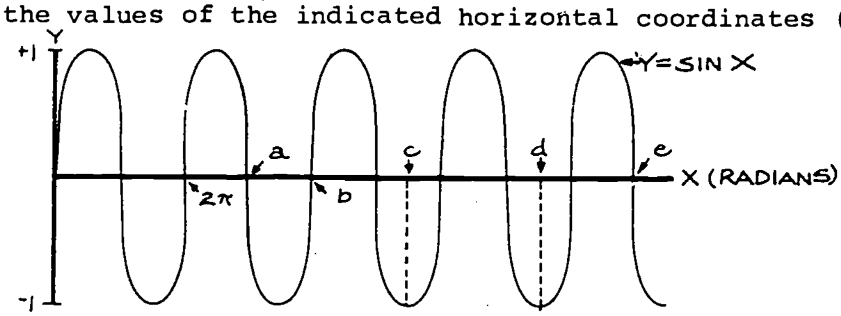
4. Use graphical addition to find the vector sum of $\vec{A}$, $\vec{B}$ and $\vec{C}$.

$\vec{A} = [4, 30°]$ $\vec{B} = [3, 80°]$ $\vec{C} = [5, 190°]$

5. Determine the values of the indicated horizontal coordinates (a, b, c, d and e).



6. a. In the figure for Problem 5, how many cycles occur between points a and e?

b. How many cycles occur between points c and d?

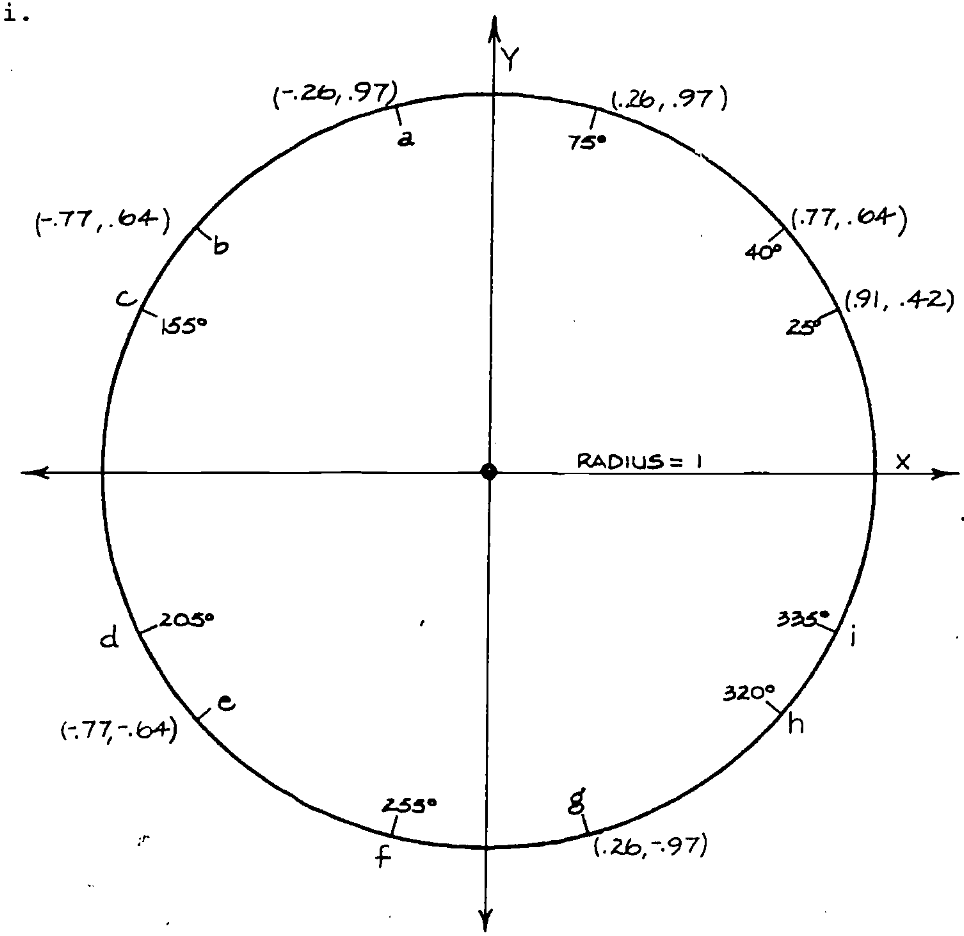
7. Complete the following sentences.

a. The sine function takes on positive values in quadrants ___?___ and ___?___.

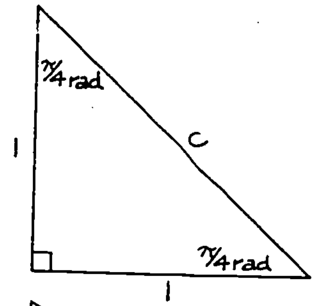
b. The cosine function takes on positive values in quadrants ___?___ and ___?___.

c. Therefore both functions are positive in quadrant ___?___.

8. The following diagram shows a unit circle marked with various angles and ordered pairs. The diagram is complete in quadrant I but in the other three quadrants some angles and ordered pairs have been left out. Supply the values for a through i.



9. a. What is the length of side c? (Use the Pythagorean Theorem.)

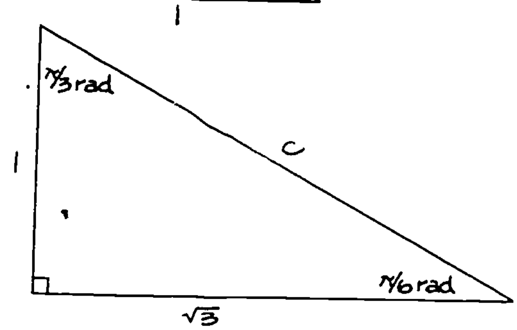


b. $\sin \frac{\pi}{4}$ rad =

c. $\cos \frac{\pi}{4}$ rad =

d. $\tan \frac{\pi}{4}$ rad =

10. a. What is the length of side C? (Use the Pythagorean Theorem.)

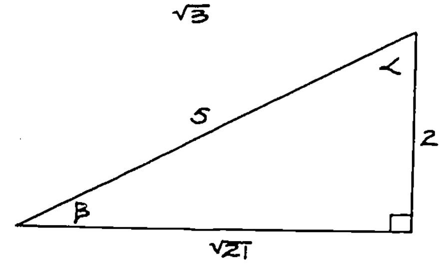


b. $\tan \frac{\pi}{3}$ rad =

c. $\cos \frac{\pi}{6}$ rad =

d. $\sin \frac{\pi}{3}$ rad =

11. Refer to the triangle at right in this question. Rationalize the denominator when necessary.



a. $\cos \beta$ = ? d. $\tan \beta$ = ?
b. $\sin \beta$ = ? e. $\cos \alpha$ = ?
c. $\tan \alpha$ = ?

12. Refer to the Trig Table in answering the following questions.

a. $\tan 43°$ = ? b. $\cos 83°$ = ? c. $\sin 1.169$ rad = ? d. $\tan .977$ rad = ?

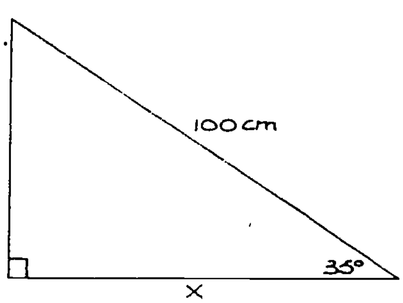
13. Find $\sin 174°$ by answering the following questions.

a. A 174° angle lies in quadrant ___________.
b. Therefore the sign of $\sin 174°$ will be (positive/negative).
c. The reference angle θ_r = ?
d. $\sin 174°$ = ?

14. What is $\cos 199°$?
15. What is $\sin 299°$?
16. What is $\sin 675°$?
17. What is $\cos \frac{11\pi}{18}$ rad ?

In each of the following problems (18-20) find the value of x with an implied uncertainty of .5. Be sure to include units.

18.



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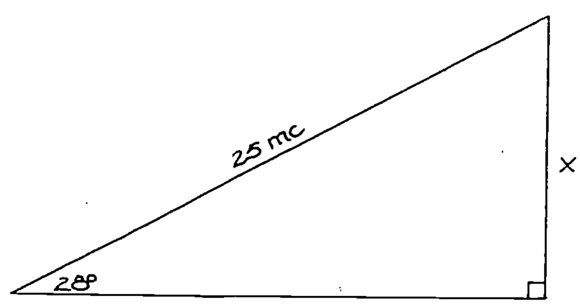

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SECTION 23: AMPLITUDE

23-1 Trigonometry and Sound

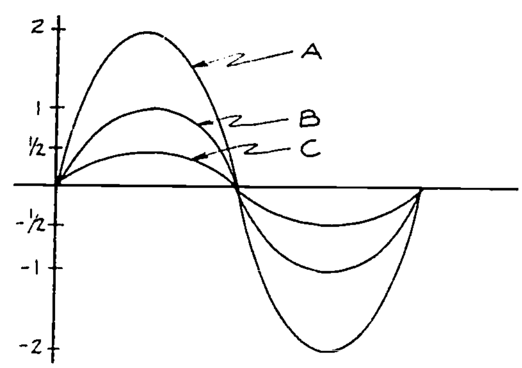
When the image of a pure sound, or tone, is displayed on an oscilloscope screen, it forms a sine curve. This fact strongly suggests that trigonometry may be used to describe and explain much of the behavior of sound waves. In fact, this is the goal of Sections 23 to 29.

An understanding of pure tones is essential for an audiologist (hearing specialist) because he uses them to test the hearing of his patients. Wave motion also plays a part in the work of optometrists and opticians, who treat vision problems. Later in this course you will see how wave motion is connected with light. For the present, we will be concerned with sound.

We will leave discussion of the physical nature of sound to the Science Text. However, we will describe what may be observed when a sound is channeled into an oscilloscope. The sound is converted to an electrical current which is then displayed as an image on the oscilloscope screen. In order for you to understand how this happens, we will first explain how the oscilloscope behaves before the sound is introduced. It very quickly swings a beam of electrons from the left to the right of the screen. It does this many times a second. Where the beam of electrons hits the screen, the screen glows. When there is no sound introduced the repeated sweeping produces a horizontal line. An incoming sound will move the beam upward or downward. The stronger the signal, the greater the movement. When the instrument produces an image for a pure tone, the result is a sine wave. The vertical coordinate is proportional to the instantaneous strength of the signal. The horizontal coordinate is related to time.

One aspect of sound that would be important to an audiologist is loudness. When testing hearing, he or she would want to know how loud a tone must be to be heard. Loudness takes a very concrete form on an oscilloscope screen. When your teacher varied the loudness of the displayed tone, the image stretched or shrank in the vertical direction. The term that describes the vertical dimension of a sine curve is amplitude. We define amplitude as the absolute value of half the distance between a peak and a valley of a sine curve. The following sine curves will illustrate the idea of amplitude. Examine them and try to think of a way in which they might be produced mathematically.

Curve A reaches a maximum of 2 and dips to a minimum of −2. The difference between the extremes is 4, and half the difference is 2. The absolute value of 2 is 2; therefore the amplitude of curve A is 2. Similarly, curve B has an amplitude of 1 and curve C has an amplitude of $\frac{1}{2}$.



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The following examples may help you see how to vary the amplitude of a sine wave mathematically. Notice that we are no longer stating the units of the angle in the sine function. From now on, <u>when units for angles are not stated, the units should be assumed to be radians</u>.

<u>EXAMPLE</u>: What is the amplitude of $y = 2 \sin x$?

<u>SOLUTION</u>:

The maximum value for the sine is 1; therefore, the maximum value for $y = 2 \sin x$ is 2 because all values of $\sin x$ are multiplied by two. Similarly, the minimum value is −2. Half the distance between $y = 2$ and $y = -2$ is 2. The absolute value of 2 is 2. In other words, the amplitude is two.

<u>EXAMPLE</u>: What is the amplitude of $y = 7 \sin x$?

<u>SOLUTION</u>:

By following the reasoning of the previous example, we find that the maximum y-value is 7 and the minimum is −7. The absolute value of half the difference between 7 and −7 is 7.

<u>EXAMPLE</u>: What is the amplitude of $y = -3 \sin x$?

<u>SOLUTION</u>:

You might want to say −3, because in the previous examples the amplitude of the curve has been the coefficient of $\sin x$. However, the definition says, "...the absolute value of half the distance..." Therefore the amplitude is $|-3|$, the absolute value of negative 3, which is 3.

<u>EXAMPLE</u>: What is the amplitude of $y = 5 \sin x + \pi$?

<u>SOLUTION</u>:

The maximum $y = 5 + \pi$

The minimum $y = -5 + \pi$

The amplitude is half the difference.

$$\frac{Y_{max} - Y_{min}}{2} = \frac{5 + \pi - (-5 + \pi)}{2} = \frac{5 + \pi + 5 - \pi}{2} = 5$$

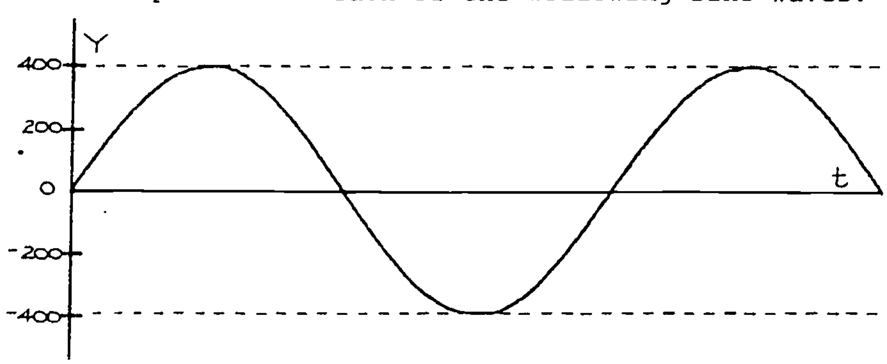
Based upon the experience gained in the above examples, we claim that the amplitude of any sine curve of the form $y = \sin x + \text{constant}$ is $|a|$, i.e., the absolute value of the coefficient of $\sin x$. It is now easy to produce sine curves of different amplitudes mathematically. For example, if we want a curve with amplitude 15, any of the following qualifies, as well as many others.

$$y = 15 \sin x \qquad y = -15 \sin x \qquad y = 15 \sin x + 21$$

PROBLEM SET 23:

Give the amplitude of each of the following sine waves.

1.



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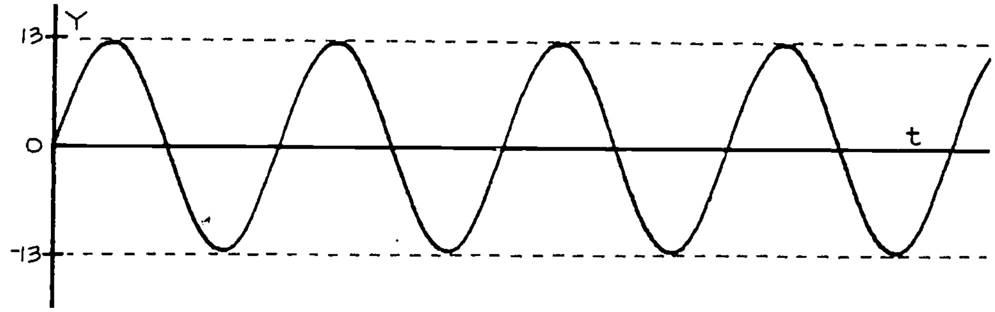

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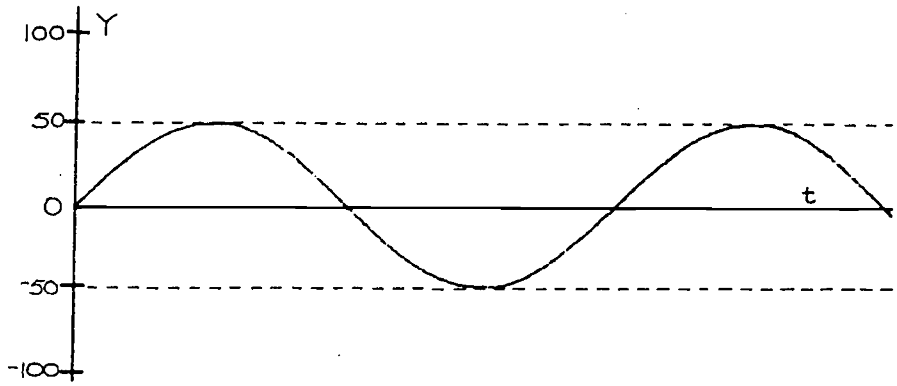

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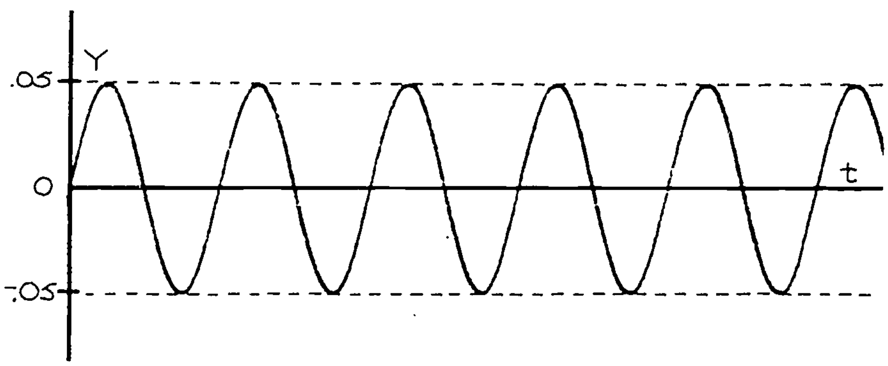

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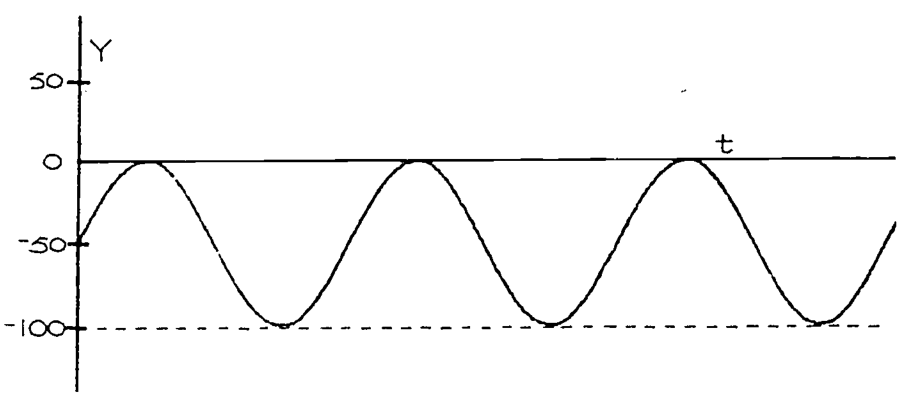

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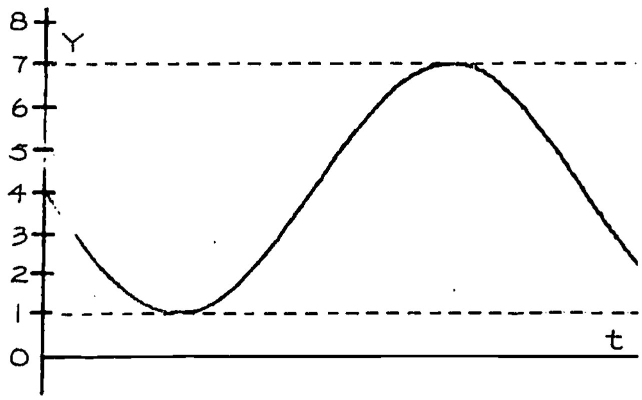
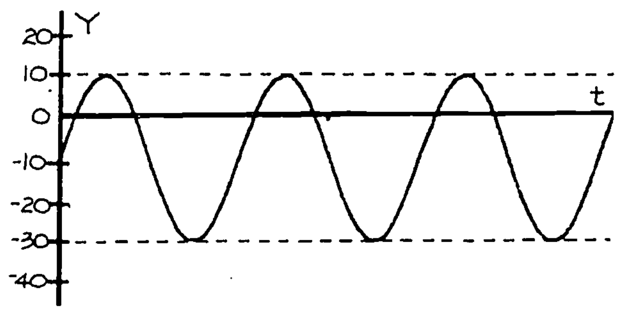
6.



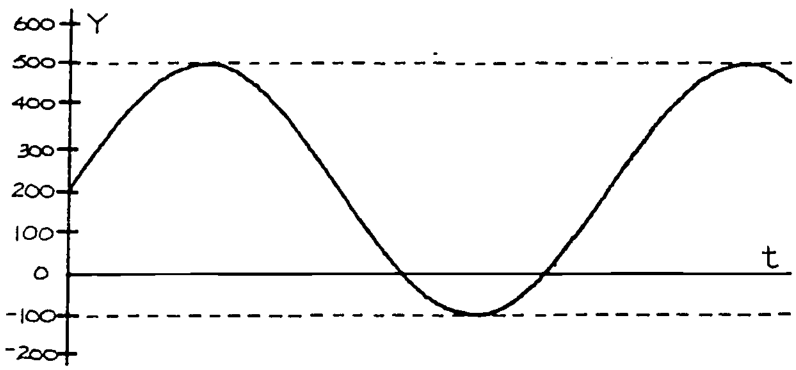
Y
8
7
6
5
4
3
2
1
0
t

7.



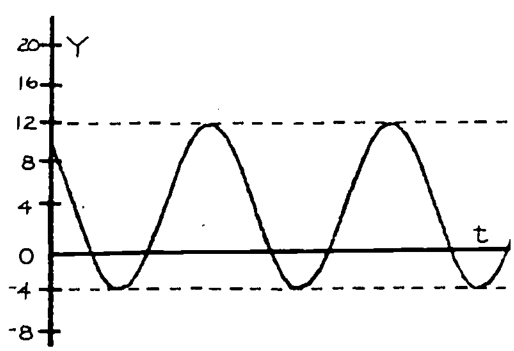
Y
20
10
0
-10
-20
-30
-40
t

8.



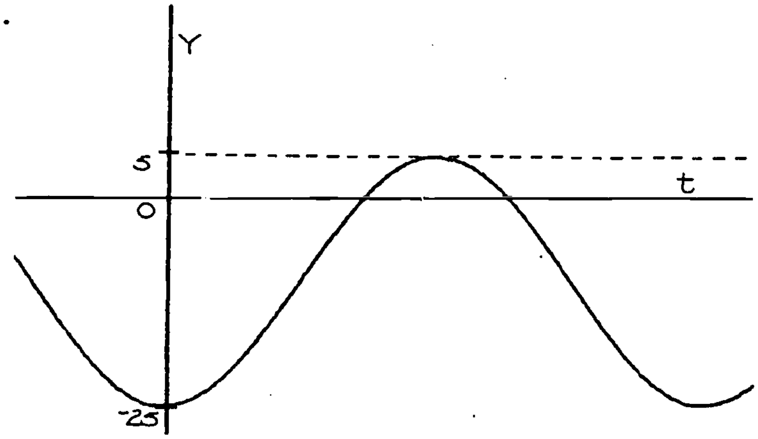
Y
600
500
400
300
200
100
0
-100
-200
t

9.



Y
20
16
12
8
4
0
-4
-8
t

10.



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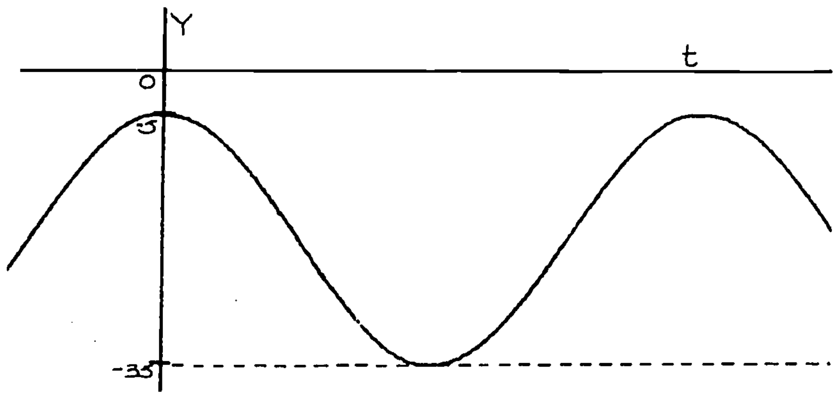

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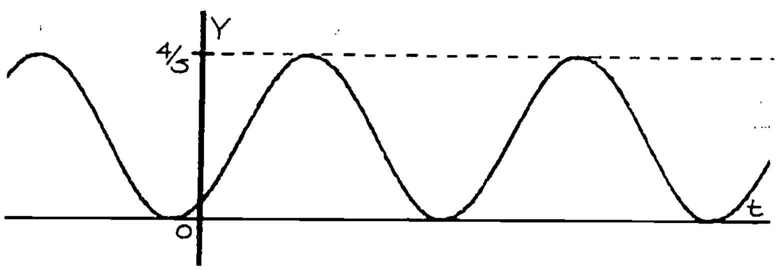


State the amplitude of the sine curves represented by the equations in Problems 13 through 22.

13. $y = 2 \sin x$

14. $y = a \sin x$

15. $y = \pi \sin x$

16. $y = -7 \sin x$

17. $y = 10^{-7} \sin x$

18. $y = 432 \sin x + 10^8$

19. $y = .07 \sin x - 23$

20. $y = (14 \times 10^{24}) \sin x - (6.024 \times 10^{23})$

21. $y = 28 - 5 \sin x$

22. $y = -100 + 100 \sin x$

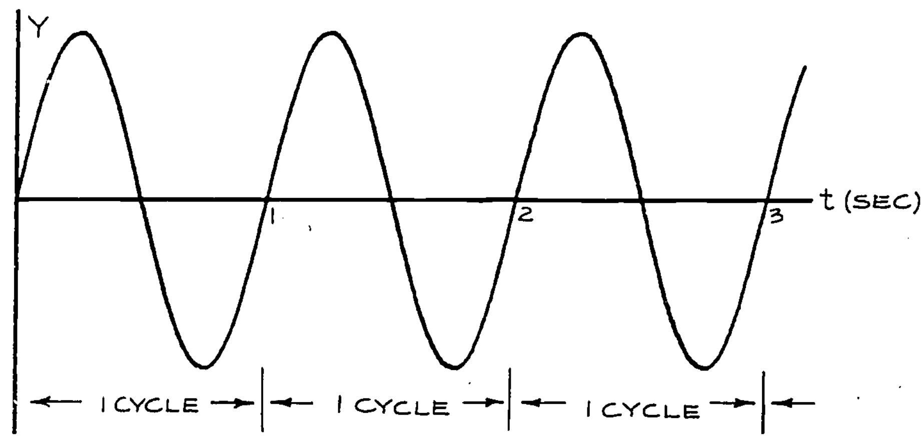
SECTION 24: FREQUENCY AND PERIOD

24-1 Frequency

When your instructor varied the pitch of the tone displayed on the oscilloscope screen, the sine curves stretched or shrank in the horizontal direction. Many terms may be used to describe this type of variation in sine curves. The two that we will use in this section are frequency and period.

We begin by defining frequency. The frequency of a sine wave is the number of cycles that occur in a unit of time. The units of frequency are cycles per unit time. Usually the unit time will be one second, in which case the units of frequency will be cycles per second (cps). Occasionally we will use other time units, such as minutes or hours. Below are some graphs of sine curves with different frequencies.

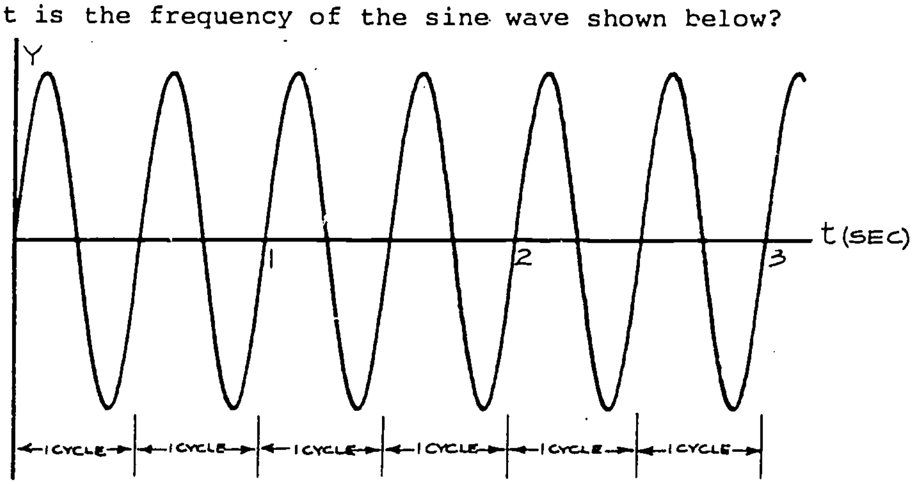
EXAMPLE: What is the frequency of the sine wave shown below?



SOLUTION:

An inspection of the graph reveals that the wave completes one cycle in one second; therefore the frequency is one cycle per second.

EXAMPLE: What is the frequency of the sine wave shown below?



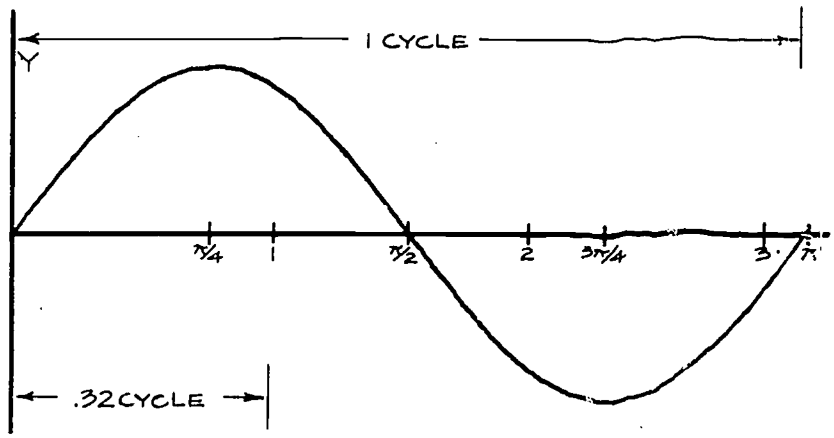
SOLUTION:

An inspection of the graph reveals that the sine wave completes two cycles each second; therefore the frequency is 2 cycles per second or 2 cps.

EXAMPLE: What is the frequency of the sine wave on the following page.

SOLUTION:

The curve completes one cycle in π sec $\approx$ 3.14 sec; therefore, the frequency is one cycle per π seconds.



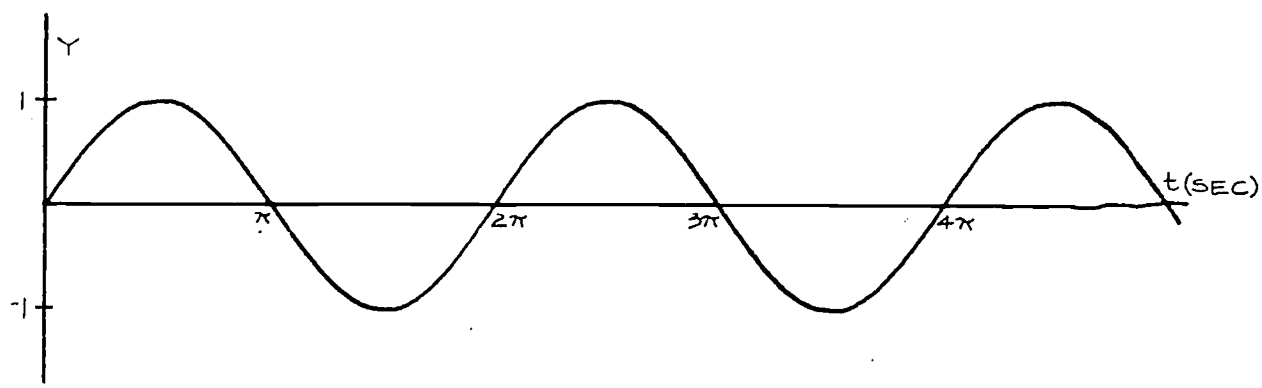
Although one cycle per π seconds is correct mathematically, it isn't stated correctly. Remember frequency must be in cycles per unit time. Our first answer states frequency in terms of the number of cycles in π seconds. To convert our first answer to the desired form we carry out the division implied by the "per" in the phrase "one cycle per π seconds." This is our second answer, mathematically equivalent to the first, but stated in standard form below.

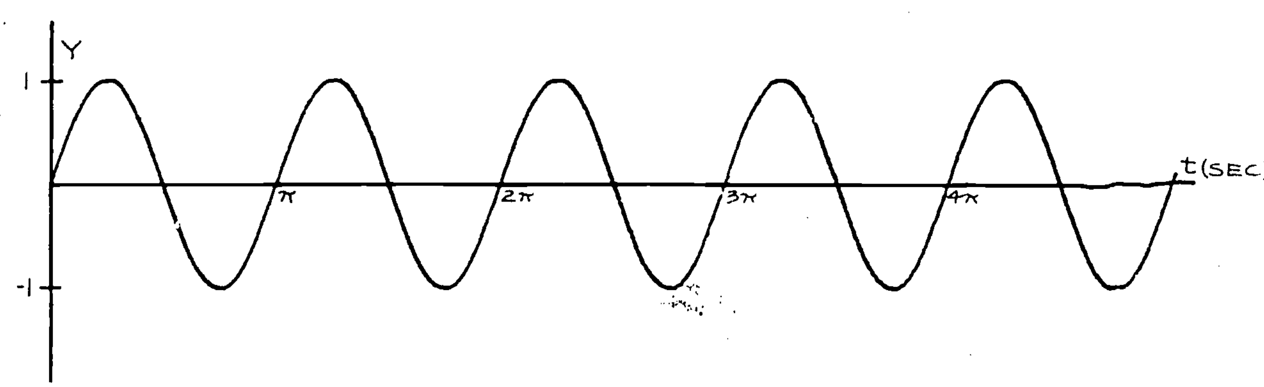
$$\frac{1 \text{ cycle}}{\pi \text{ seconds}} \approx \frac{1}{3.14} \frac{\text{cycle}}{\text{second}}$$

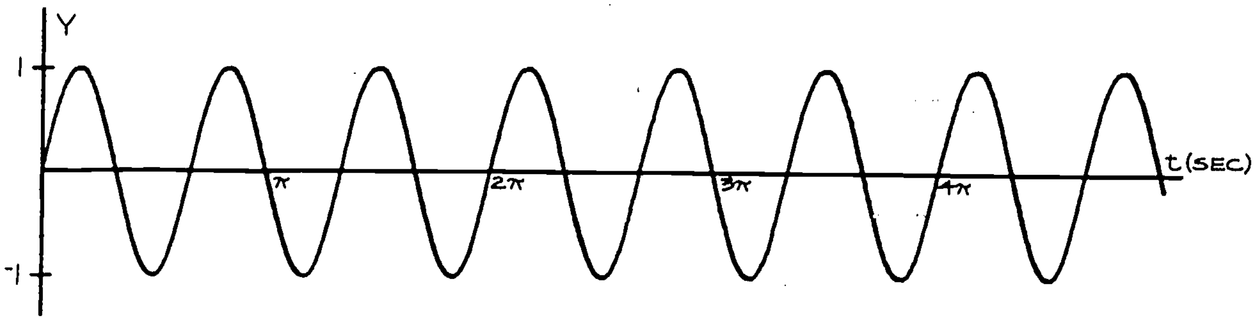
$$\approx .32 \frac{\text{cycle}}{\text{second}}$$

24-2 How May the Frequency Be Varied Mathematically?

Below and on the next page are the graphs of y = sin t, y = sin 2t and y = sin 3t. Observe that by varying the coefficient of t, the frequency may be varied.







First, notice that the frequency is directly related to the coefficient of t. If the coefficient is increased, the frequency is increased. The higher the frequency, the larger the coefficient. Notice that when the coefficient is doubled, the frequency doubles. When it is tripled, the frequency triples.

Second, notice that the coefficient of t is not numerically equal to the frequency.

SINE WAVE	FREQUENCY
$y = \sin t$	$\frac{1 \text{ cycle}}{2\pi \text{ seconds}} \approx .16$ cps
$y = \sin 2t$	$\frac{2 \text{ cycles}}{2\pi \text{ seconds}} \approx .32$ cps
$y = \sin 3t$	$\frac{3 \text{ cycles}}{2\pi \text{ seconds}} \approx .48$ cps

Examine the pattern in the table. It reveals the relationship between the frequency of a curve and the coefficient of t. It is

$$\text{frequency} = \frac{\text{coefficient of } t}{2\pi}$$

This is a very important relationship. You should become familiar with it. It may be used to find the frequency if the coefficient is known. Or, it may be used to determine the coefficient needed to produce a curve of a given frequency as illustrated in the example below.

EXAMPLE:

Find the value of b so that the sine curve given by $y = \sin bt$ will have a frequency of 7cps.

SOLUTION:

The relationship between frequency and the coefficient of t is given by the equation

$$\text{frequency} = \frac{\text{coefficient of } t}{2\pi}$$

The coefficient of t is b and the desired frequency is 7 cps. We substitute into the equation and solve for b.

$$7 = \frac{b}{2\pi}$$

$$7\,(2\pi) = \frac{b}{2\pi} \cdot 2\pi$$

$$14\pi = b$$

Therefore the equation for the sine curve is $y = \sin 14\pi t$.

You may have observed during the oscilloscope demonstration that tones with higher frequencies have higher pitches. Frequency is a measure of pitch. The frequency of middle C is 256 cps. The frequency of high C is 512 cps. Each musical note has its corresponding frequency.

24-3 Period

Period is another term that is used to describe the horizontal dimensions of a sine curve.

The period of a sine wave is the time required to complete one cycle. Refer back to the graphs of y = sin t, y = sin 2t and y = sin 3t. The time required for y = sin t to complete one cycle is $2\pi \approx 6.28$ seconds. The time required for y = sin 2t to complete one cycle is $\pi \approx 3.14$ seconds.

Mathematically, period is the reciprocal of frequency and period is expressed in units of time per cycle. Usually time will be measured in seconds, in which case period will be measured in seconds per cycle. Often the "per cycle" is omitted, and the period is given merely as a length of time. For example, we might say that the period of a curve is .1 second or .5 minute.

EXAMPLE: A particular sound has a frequency of 100 cps. What is its period?

SOLUTION:

Period is the reciprocal of frequency.

$$\text{period} = \frac{1}{\text{frequency}}$$

$$\text{period} = \frac{1}{100 \text{ cps}}$$

$$\text{period} = .01 \ \frac{\text{second}}{\text{cycle}}$$

We could also give the answer as period = .01 second.

24-4 Formulas, Notation and Conventions

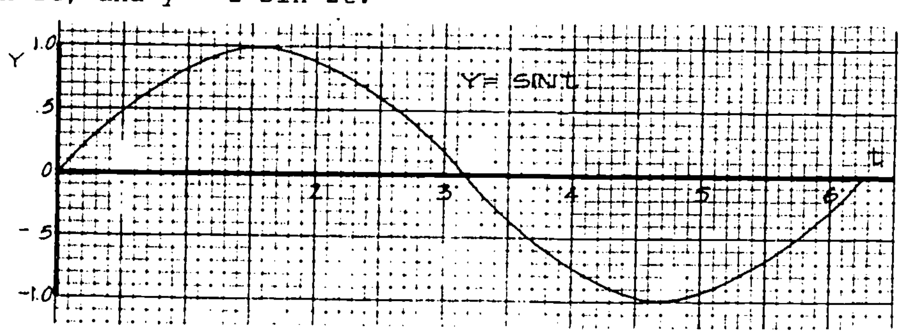
We will use the letter "f" to represent frequency. The capital letter "T" is used to represent period. The formula that relates the two is $\frac{1}{f} = T$ or, in words, "The reciprocal of the frequency is the period."

For the coefficient of t we will use the letter "b", i.e., y = sin bt. Finally b is related to f by the equation $f = \frac{b}{2\pi}$ or, in words, "The frequency of a sine curve is the coefficient of t divided by 2π."

Since $2\pi f = b$, we can substitute $2\pi f$ for b in the equation y = sin bt to get $y = \sin 2\pi ft$, an equation which is sometimes easier to use.

24-5 Amplitude, Frequency, Period, Loudness and Pitch

The following four graphs show the distinction between amplitude on the one hand and period and frequency on the other. The graphs are for y = sin t, y = 2 sin t, y = sin 2t, and y = 2 sin 2t.



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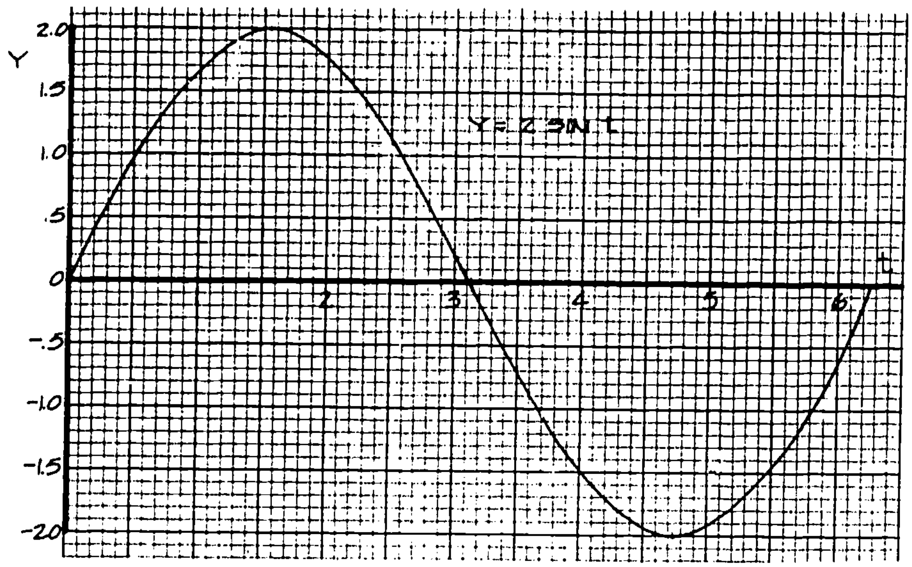
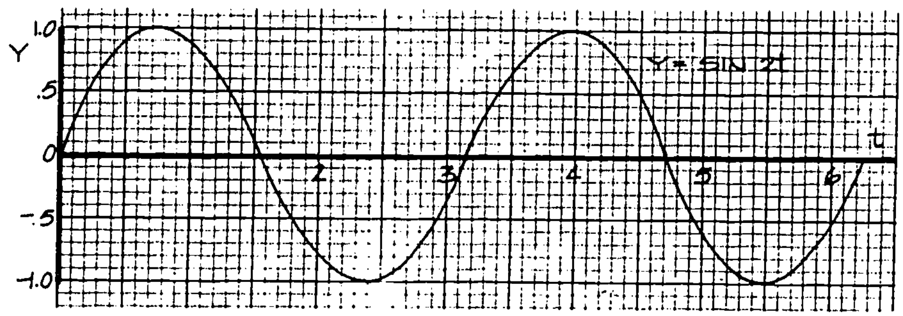
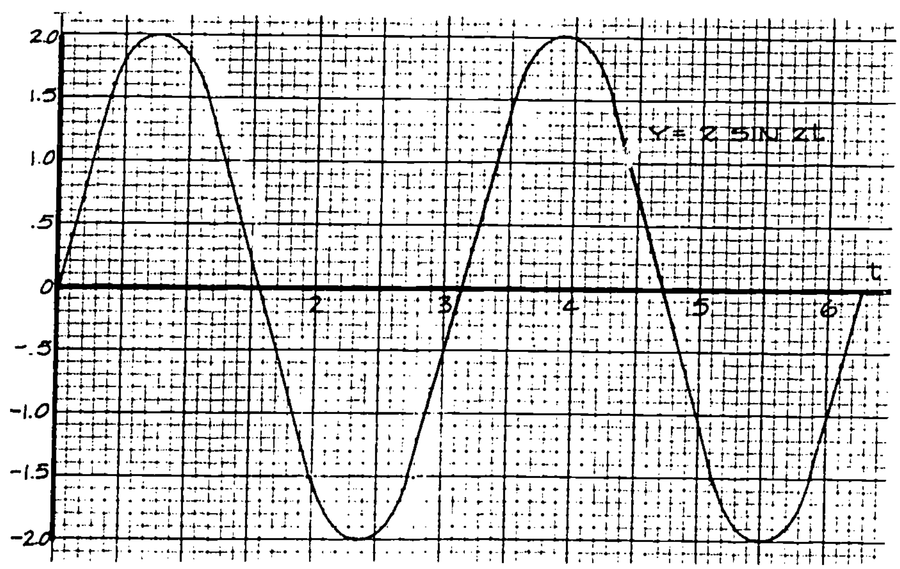

Y
Y = 2 SIN t
t


Y
Y = SIN 2t
t


Y = 2 SIN 2t
t

If these sine waves represent sounds, then y = sin 2t and y = 2 sin 2t are sounds of higher pitch because they have higher frequencies. Of the two, 2 sin 2t is louder because it has a greater amplitude. The functions y = sin t and y = 2 sin t are sounds of lower pitch because they have lower frequencies. Since y = 2 sin t has a greater amplitude it represents the louder sound.

PROBLEM SET 24:

1. Identify whether the following expressions are statements of frequency (f, $\frac{\text{cycles}}{\text{time unit}}$) or period (T, $\frac{\text{time}}{\text{cycle}}$).

a. $14\ \frac{\text{minutes}}{\text{cycle}}$

b. $27\ \frac{\text{cycles}}{2\pi\ \text{microseconds}}$

c. $\frac{\frac{1}{33}\ \text{fortnight}}{\text{cycle}}$

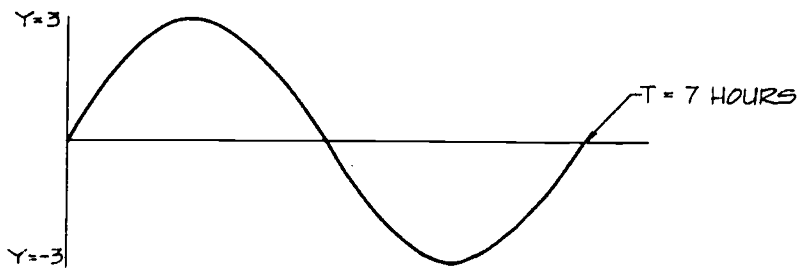
d. $2\pi\ \frac{\text{fortnights}}{\text{cycle}}$

e. $1\ \frac{\text{cycle}}{\sqrt{2}\ \text{days}}$

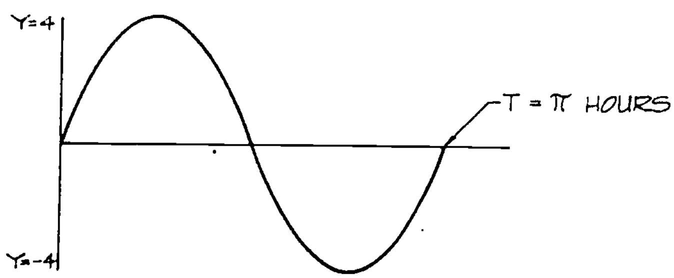
f. $1\ \frac{\text{lunar month}}{\text{cycle}}$

State the frequency and the period of the sine curves in Problems 2 through 9.

2.



3.



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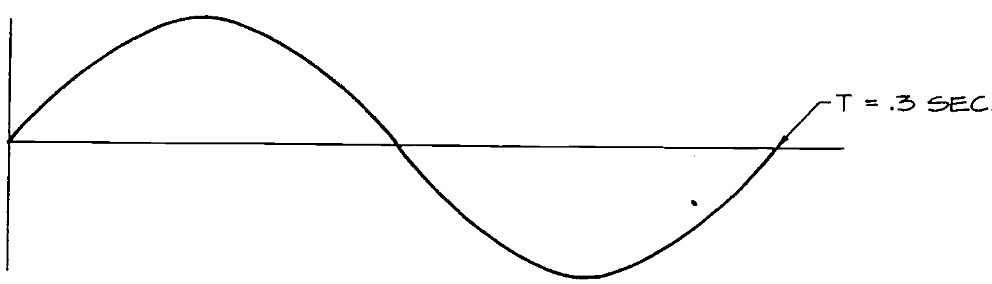

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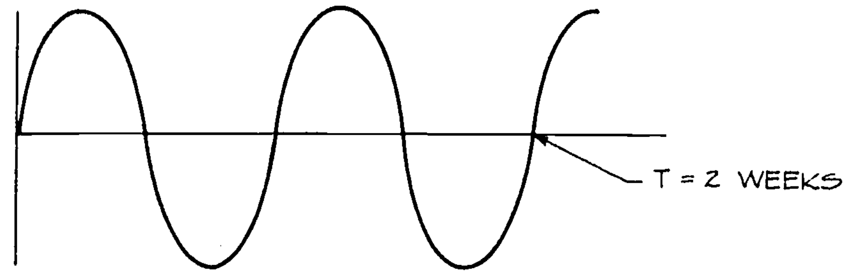

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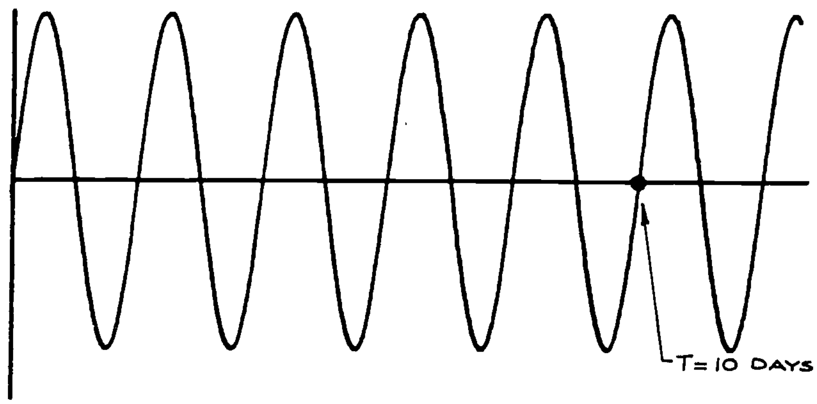

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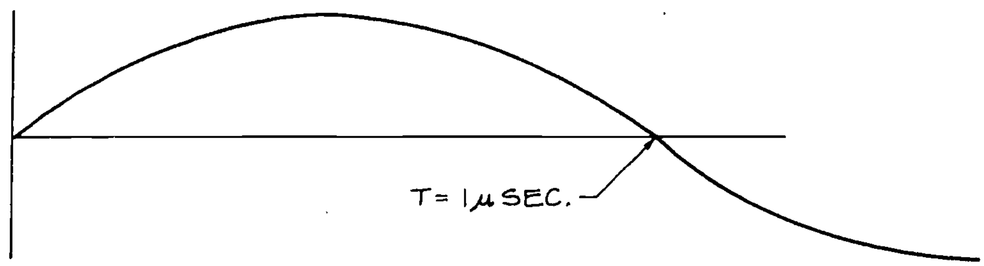

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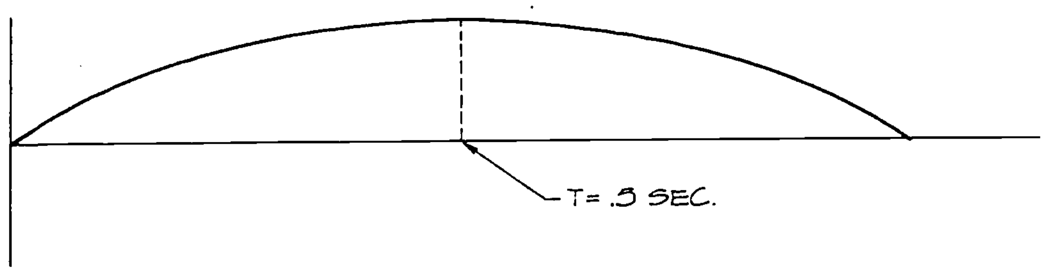
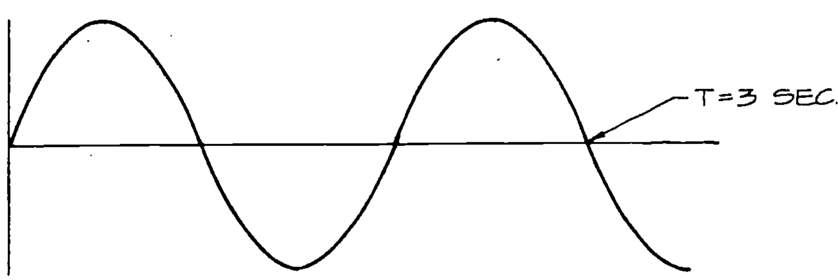


9.



10. a. Complete the table.

t (sec)	sin t	sin $\frac{1}{2}$t
0	?	?
$\frac{\pi}{4}$	?	SKIP
$\frac{\pi}{2}$	?	?
$\frac{3\pi}{4}$	?	SKIP
π	?	?
$\frac{5\pi}{4}$	?	SKIP
$\frac{3\pi}{2}$	?	?
$\frac{7\pi}{4}$	?	SKIP
2π	?	?

b. Sketch y = sin t and y = sin $\frac{1}{2}$t on the same graph.

c. State the frequency and period of each sine curve.

11. a. Complete the tables.

t (sec)	sin πt
0	?
.5	?
1.0	?
1.5	?
2.0	?
2.5	?
3.0	?

t (sec)	sin 2πt
0	?
.25	?
.50	?
.75	?
1.00	?
1.25	?
1.50	?

b. Sketch y = sin πt and y = sin 2πt on the same graph.

c. State the frequency and period of each sine curve.

State the frequency and period of each of the equations in Problems 12 through 17. The horizontal axis is in seconds. Include units.

12. y = sin t
13. y = sin 33t
14. y = sin πt
15. y = sin 16πt
16. y = sin 3πt
17. y = sin $\frac{1}{2}$t

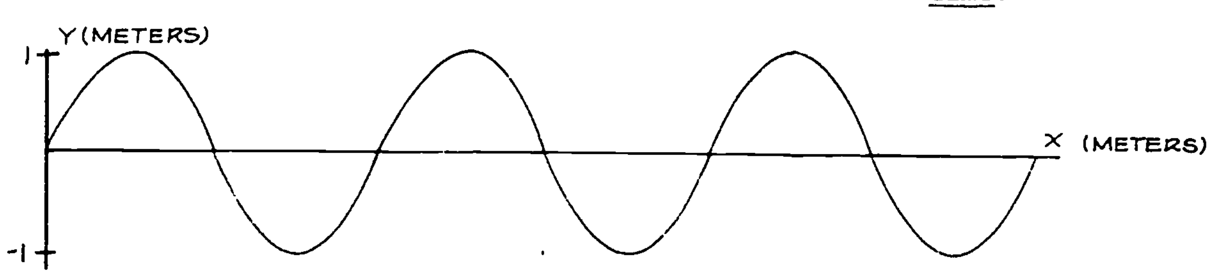
18. Identify the sine function in Problems 12 through 17 that represents the sound with the highest pitch.

SECTION 25: TRAVELING WAVES AND WAVELENGTH

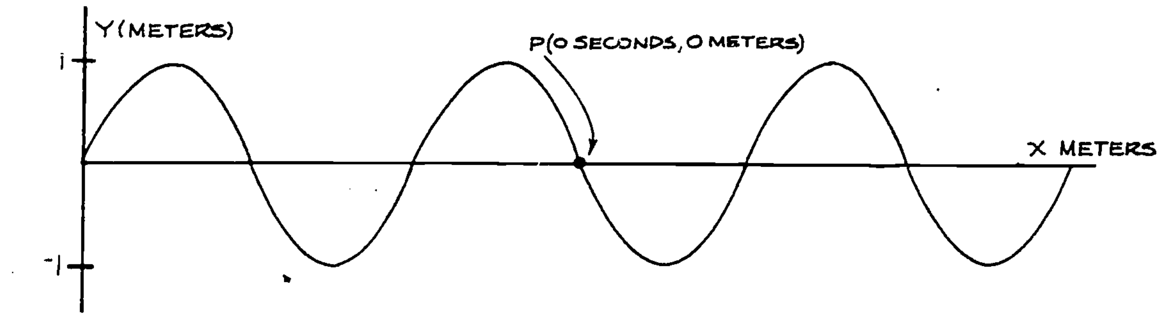
25-1 Traveling Waves--Points of View

Real waves such as sound waves or light waves don't just sit around waiting patiently to be poked, prodded and inspected as the sine curves in this book do. They move, just as waves in the ocean do. In order to stop or freeze such a traveling wave, we can imagine taking a "snapshot" of the wave with a high-speed camera. Such a snapshot is shown below. In order to make things simple, we have assumed that the amplitude of the wave is 1 meter.

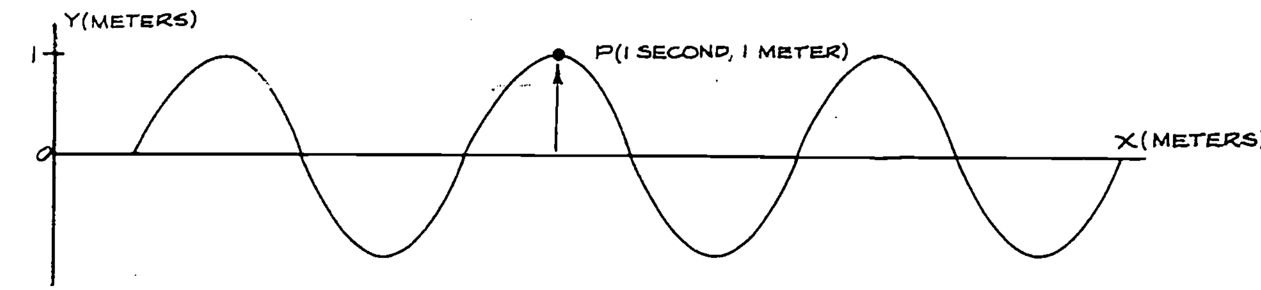
How does this graph differ from those of the last sections? The answer is that the horizontal axis is scaled in units of distance, in this case meters. In all the previous graphs, the horizontal axis was scaled in units of time.



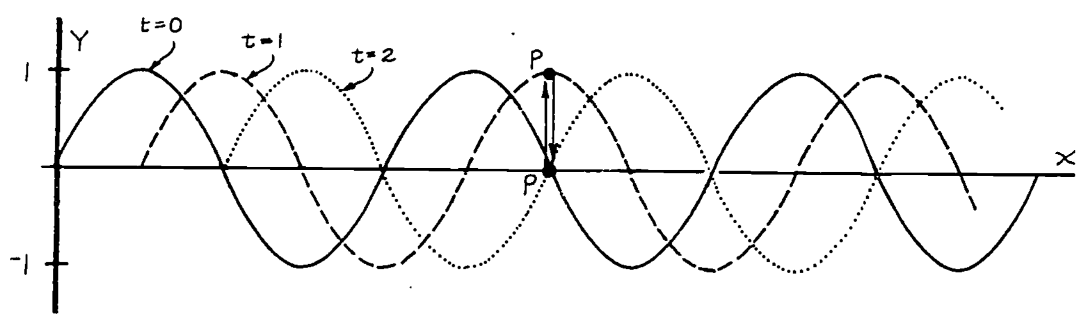
However, there is an easy way to obtain a graph with the horizontal axis scaled in in time. To do this, we display the snapshot again and imagine that there is a cork floating on the curve at point P. We will assume that the snapshot was taken at time t = 0 seconds. As you can see, the cork (point P) is located on the x-axis. The coordinate of P is 0. This generates an ordered pair (0 seconds, 0 meters).



Below is a snapshot of the same wave taken 1 second later. Cork P is now riding the crest of a wave. This snapshot generates our second ordered pair (1 second, 1 meter).



Below is a set of three snapshots taken at t = 0, t = 1 and t = 2 seconds that shows the behavior of cork P as the waves pass. As the wave travels from left to right, the cork bobs up and down. The arrows show the motion of the cork.

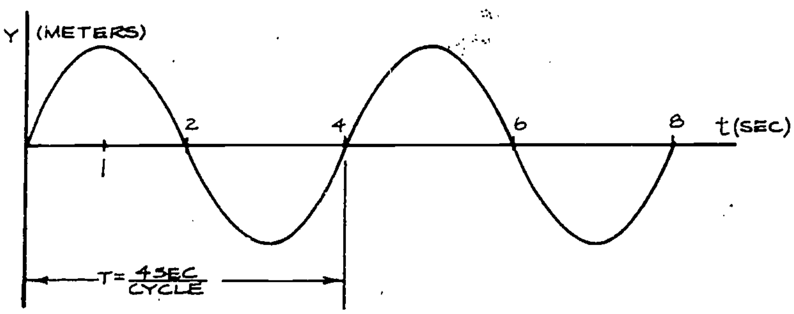


At t = 3 the cork will float down to y = -1 and at t = 4 it will return again to y = 0. This pattern will continue indefinitely as the wave travels past.

A table function may be made of the vertical displacement of the cork as a function of time.

t (sec)	y (meters)
0	0
1	1
2	0
3	-1
4	0

When these points are graphed and connected with a smooth curve, a sine curve is obtained.

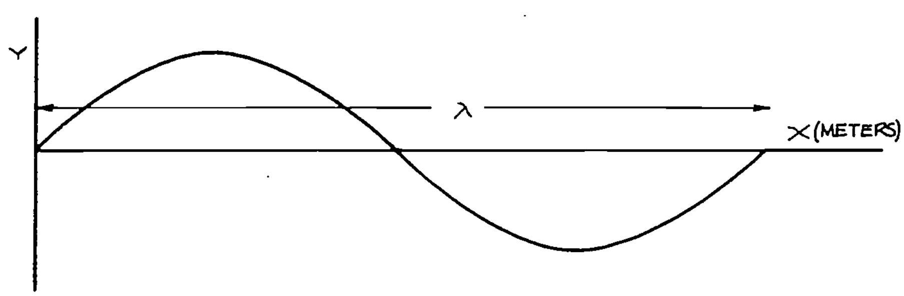


An examination of the curve reveals that T = 4 seconds per cycle and $f = \frac{1}{4}$ cycle per second. In terms of our cork analogy this means that the cork would bob up and down and back up to its starting position.

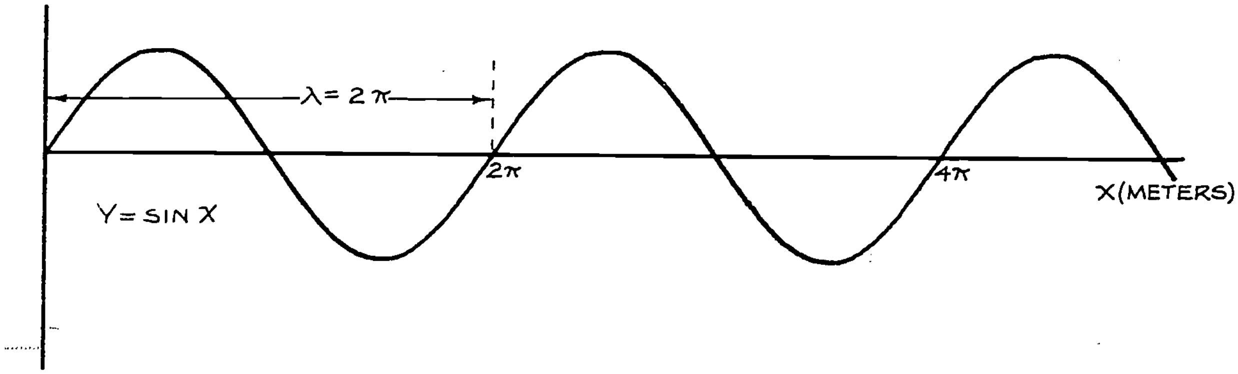
25-2 Wavelength

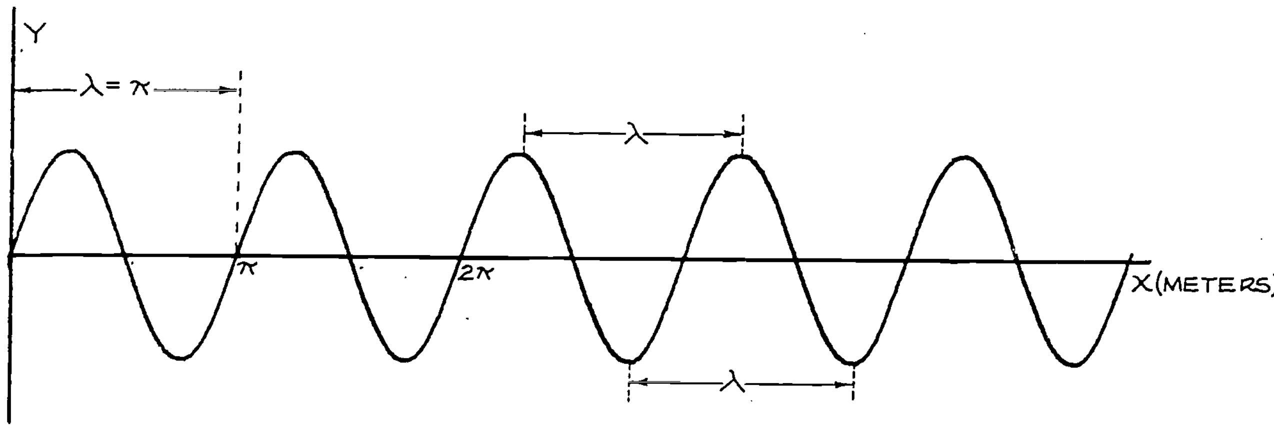
Wavelength is the length of one wave. When a sine curve is considered to be a snapshot of an actual traveling wave, the wavelength may be determined from the graph.

The units of wavelength are length per cycle. (You will commonly see wavelength given simply in a unit of length, which implies length per cycle.) Wavelength is generally symbolized by λ (the Greek letter "lambda").



The wavelength of the curve y = sin x is 2π length units per cycle. The wavelength of y = sin 2x is π length units per cycle.





Both wavelength and period measure the length of a wave along a horizontal axis. But the two do not measure the same quantity. Period is measured in units of time per cycle, while wavelength is measured in units of length per cycle.

There are other ways to determine wavelength or period from a graph besides the ones we have shown you so far. Wavelength may be considered to be the distance between crests, or the distance between troughs as shown in the diagram above. Similarly, period may be considered to be the time between two successive crests or troughs. These methods of determining wavelength or period are often more convenient to use in practice than the method we demonstrated earlier.

25-3 Formulas and Equations--A Summary

Recall that for $y = a \sin bt$

$$f = \frac{b}{2\pi} \quad \text{and} \quad f = \frac{1}{T}$$

When $\frac{1}{T}$ is substituted for f in the first equation, we obtain

$$\frac{1}{T} = \frac{b}{2\pi}$$

or, equivalently

$$T = \frac{2\pi}{b}$$

These formulas all apply to sine functions of time, that is, functions of the form $y = a \sin bt$. When we are dealing with sine functions of distance, we will use the equation $y = a \sin cx$, where x designates distance units. The constant c plays the same role in the distance equation as b does in the time equation. There is another parallel. The quantity λ (wavelength) on a distance graph measures the same thing at T (period) on the time graph. Therefore, we can obtain a set of new formulas for distance graphs by taking the ones listed on page 13 and replacing b by c and T by λ. This gives

$$\lambda = \frac{2\pi}{c}$$

or, equivalently

$$c = \frac{2\pi}{\lambda}$$

EXAMPLE: Determine the wavelength of the function $y = \sin 2\pi x$.

SOLUTION:

The function is of the form $y = \sin cx$, with $c = 2\pi$. We use the formula

$$\lambda = \frac{2\pi}{c}$$

We substitute 2π for c and obtain

$$\lambda = \frac{2\pi}{2\pi}$$

$$\lambda = 1 \ \frac{\text{unit of length}}{\text{cycle}}$$

In addition to amplitude, we have mentioned three other properties of waves. These three are summarized below, so that you may refer back to this table.

	Symbol	Horizontal axis	Form of equation	Units
period	T	time	$y = a \sin bt$	$\frac{\text{units of time}}{\text{cycle}}$
frequency	f	time	$y = a \sin bt$	$\frac{\text{cycles}}{\text{units of time}}$
wavelength	λ	distance	$y = a \sin cx$	$\frac{\text{units of length}}{\text{cycle}}$

25-4 The Speed of a Wave

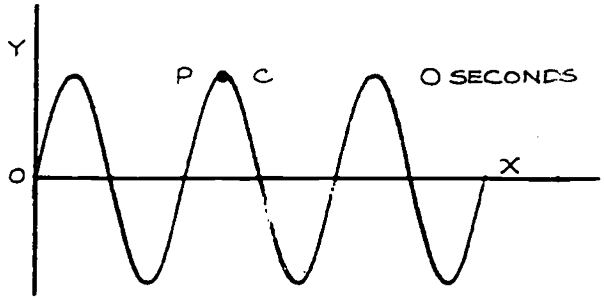
Suppose we let d = distance, s = speed and t = time. Then,

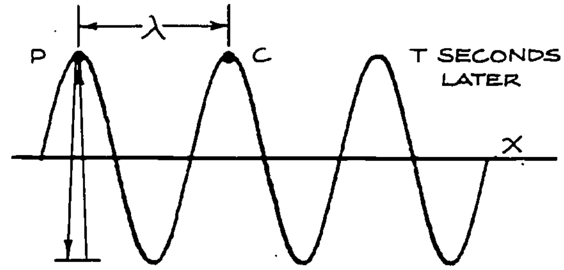
$$d = st$$

Distance equals speed times time. This is a familiar formula. We may use this formula to determine the speed (s) with which a particular wave is moving. One way to apply the formula would be to measure the distance (d) that a particular wave crest moves in a particular time (t) as measured by a stopwatch. Then the speed (s) could be calculated ($\frac{d}{t} = s$). However there is a more elegant method.

Consider a cork (P) bobbing on a wave as shown at the top of the following page. The snapshot on the right was taken one period (T seconds) after the one on the left. In that time the cork has bobbed down and up once and the wave has moved one wavelength (λ meters) to the right. In other words, crest C has moved λ meters in T seconds. These two figures may be used to calculate the speed at which the wave is moving. We let $d = \lambda$ and $t = T$ in the formula $\frac{d}{t} = s$ to obtain

$$\frac{\lambda}{T} = s$$





or, when the relation $f = \frac{1}{T}$ is used

$$\lambda\left(\frac{1}{T}\right) = s$$
$$\lambda f = s$$

Including the dimensions, you can see that the speed s will have the proper units.

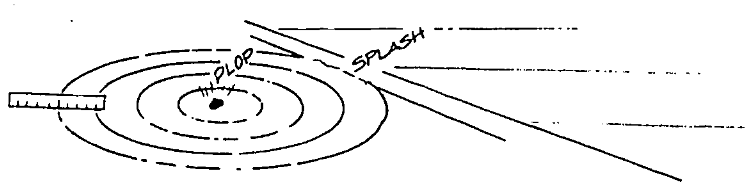
$$\lambda \frac{\text{meters}}{\text{cycle}} \cdot f \frac{\text{cycles}}{\text{second}} = s \frac{\text{meters}}{\text{second}}$$

By use of these formulas the speed of a wave may be calculated if the wavelength (λ) and either the period (T) or frequency (f) is known.

This relationship between speed, wavelength and frequency is useful in studying waves, and you will use it in Laboratory Activity 23 to determine the speed of sound.

EXAMPLE:

Elmo drops a stone into a swimming pool and measures the distance between two successive waves with a meter stick. This distance is the wavelength, and is 40 centimeters per cycle. Elmo counts the number of waves that hit the side of the pool to determine the frequency. He computes that one wave hits the side every two seconds, so the frequency is $\frac{1}{2}$ cycle per second. What is the speed of the waves?



SOLUTION:

We are given that

$$\lambda = 40 \frac{\text{cm}}{\text{cycle}}$$
$$f = \frac{1}{2} \frac{\text{cycle}}{\text{sec}}$$

We know the formula for speed (s) in terms of λ and f is

$$s = \lambda f$$

Therefore,

$$s = 40 \frac{\text{cm}}{\text{cycle}} \cdot \frac{1}{2} \frac{\text{cycle}}{\text{sec}}$$
$$= 20 \frac{\text{cm}}{\text{sec}}$$

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PROBLEM SET 25:

1. Match the following expressions.

a. lambda	f
b. period	λ
c. wavelength	
d. frequency	T

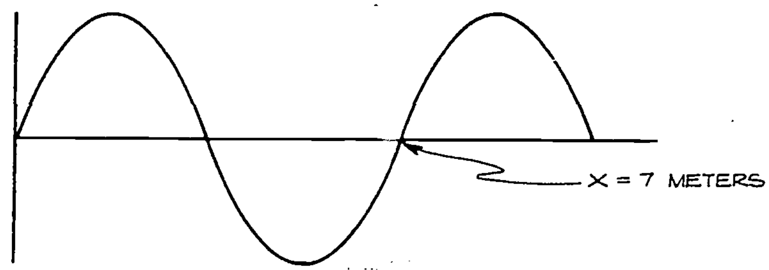
2. Identify whether the following expressions are statements of frequency (f, $\frac{\text{cycles}}{\text{unit time}}$), period(T, $\frac{\text{unit of time}}{\text{cycle}}$) or wavelength ($\lambda$, $\frac{\text{unit of length}}{\text{cycle}}$).

a. $\frac{\text{cycles}}{\text{sec}}$ c. $\frac{\text{hours}}{\text{cycle}}$ e. $\frac{\text{hectometers}}{\text{cycle}}$ g. $\frac{\text{cm}}{\text{cycle}}$

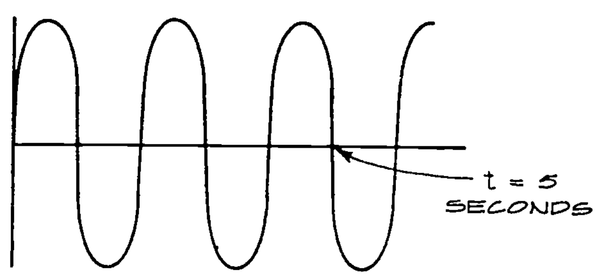
b. $\frac{\text{meters}}{\text{cycle}}$ d. $\frac{\text{fortnights}}{\text{cycle}}$ f. $\frac{\text{cycles}}{\text{microsecond}}$

State the wavelength or period of the sine curves in Problems 3 through 7. Specify either wavelength or period, whichever is appropriate. <u>Include units</u>.

3.



4.



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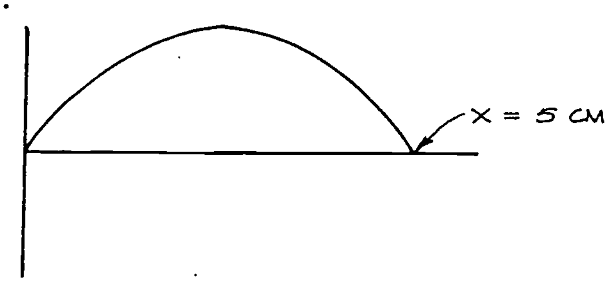
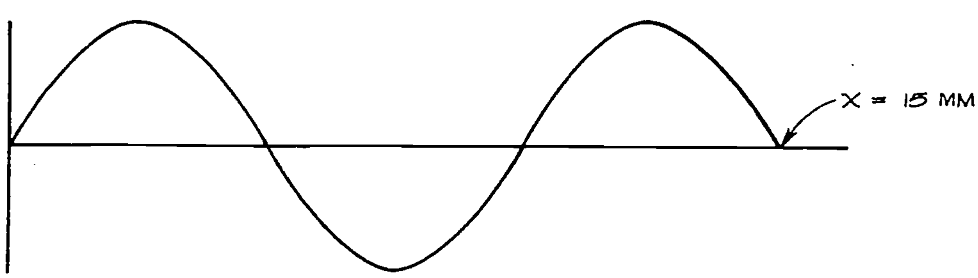
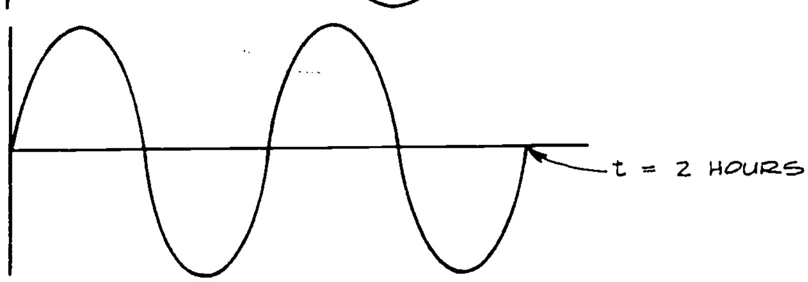


6.



7.



State the wavelength λ for each of the sine curves described by the equations in Problems 8 through 11. The horizontal axis is in meters. Include units.

8. $y = \sin x$ 9. $y = \sin 14x$ 10. $y = \sin \frac{x}{\pi}$ 11. $y = \sin 3\pi x$

State the speed of the waves for each situation described in Problems 12 through 14. Include units.

12. $\lambda = 300 \frac{\text{m}}{\text{cycle}}$ $f = 140 \frac{\text{cycles}}{\text{sec}}$

13. $\lambda = 16 \frac{\text{cm}}{\text{cycle}}$ $f = 3 \times 10^4 \frac{\text{cycles}}{\text{sec}}$

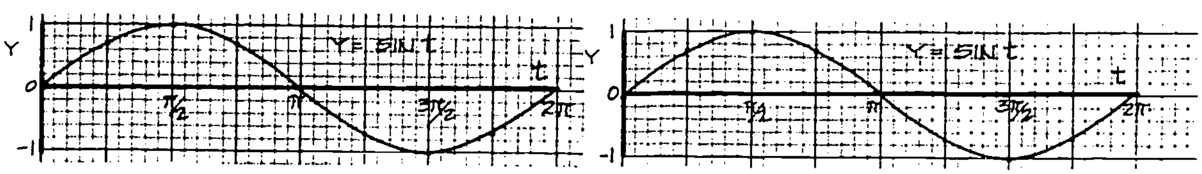
14. $\lambda = 1 \times 10^{-3} \frac{\text{mm}}{\text{cycle}}$ $f = 3 \times 10^6 \frac{\text{cycles}}{\text{sec}}$

SECTION 26: ADDING WAVES AND PHASE

So far the sine waves we have considered are the type associated with pure musical tones. However, most of the sounds you hear are not pure tones. Instead they are combinations of many tones, and the waves associated with them are more complicated than a simple sine wave. In the next three sections we will explore the waveforms associated with combinations of pure tones.

What happens when two waves meet and combine? They simply add together at every point. In this section we will consider the addition of waves that have the same period and frequency. In Section 27 we will be concerned with the addition of waves with different periods and frequencies.

Consider the two sine curves below; the equation of each is $y = \sin t$.



The y-coordinate of each sine curve at $t = 0$ is zero. Therefore, at $t = 0$ the sum of the waves is zero. When $t = \frac{\pi}{4}$, each has a y-coordinate of approximately .71. The sum of the y-coordinates is $.71 + .71 = 1.42$. When $t = \frac{\pi}{2}$, the y-coordinate of each curve is 1.0. The sum of these coordinates is $1.0 + 1.0 = 2.0$.

We may continue the process for any t-coordinate. If we choose intervals of $\frac{\pi}{4}$, we obtain the values shown in the table on the following page.

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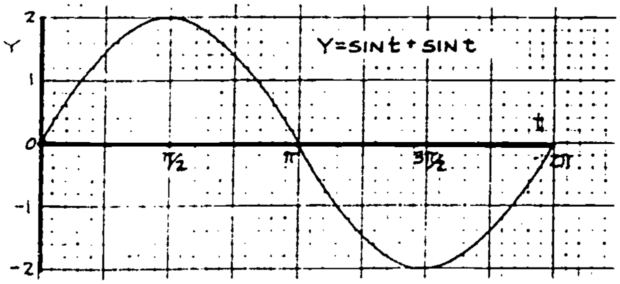
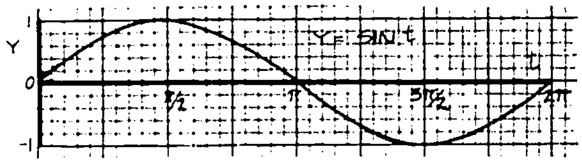
A graph of the sum of the waves is shown below.

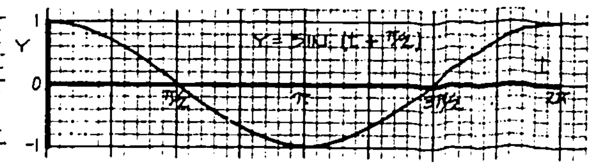


t	sin t	sin t	sin t + sin t
0	.00	.00	.00
$\frac{\pi}{4}$	.71	.71	1.42
$\frac{\pi}{2}$	1.00	1.00	2.00
$\frac{3\pi}{4}$	.71	.71	1.42
π	.00	.00	.00
$\frac{5\pi}{4}$	-.71	-.71	-1.42
$\frac{3\pi}{2}$	-1.00	-1.00	-2.00
$\frac{7\pi}{4}$	-.71	-.71	-1.42
2π	.00	.00	.00

The sum is a sine wave with the same period (2π time units) but twice the amplitude (2 distance units) of the individual waves. The two waves, y = sin t, are said to be in phase. We say that two waves are in phase whenever the peaks of the two curves have the same horizontal coordinates.

Now consider the sum of the sine curves y = sin t and y = sin $(t + \frac{\pi}{2})$. The graph of y = sin t is on the left below; the graph of y = sin $(t + \frac{\pi}{2})$ is on the right. Notice that the $\frac{\pi}{2}$ term in sin $(t + \frac{\pi}{2})$ moves the sine curve $\frac{\pi}{2}$ units to the left.





Also note that y = sin $(t + \frac{\pi}{2})$ has the same period as y = sin t; the period is 2π time units for both curves.

The sum of sin t and sin $(t + \frac{\pi}{2})$ when t = 0 is the sum of sin 0 and sin $\frac{\pi}{2}$.

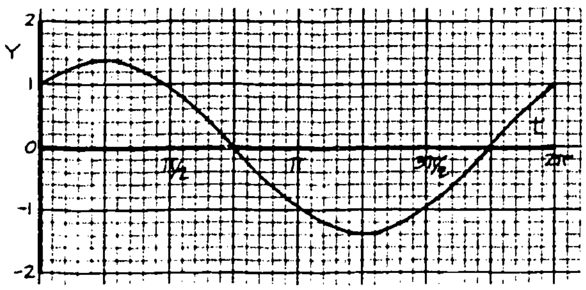
$$\sin 0 + \sin \frac{\pi}{2} = 0 + 1$$
$$= 1$$

The sum of sin t and sin $(t = \frac{\pi}{2})$ when $t = \frac{\pi}{4}$ is

$$\sin \frac{\pi}{4} + \sin (\frac{\pi}{4} + \frac{\pi}{2}) = \sin \frac{\pi}{4} + \sin \frac{3\pi}{4}$$
$$\approx .71 + .71$$
$$\approx 1.42$$

These sums and the sums for other values of t are listed in the table on the following page.

The sum $y = \sin t + \sin (t + \frac{\pi}{2})$ is represented by the next graph.



$y = \sin t + \sin (t + \frac{\pi}{2})$

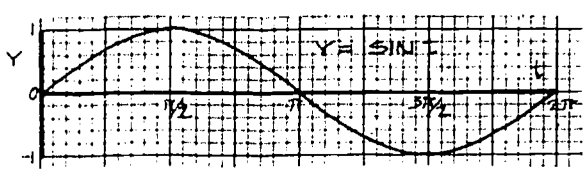
t	$\sin t$	$\sin (t + \frac{\pi}{2})$	$\sin t + \sin (t + \frac{\pi}{2})$
0	.00	1.00	1.00
$\frac{\pi}{4}$	.71	.71	1.42
$\frac{\pi}{2}$	1.00	.00	1.00
$\frac{3\pi}{4}$	.71	-.71	.00
π	.00	-1.00	-1.00
$\frac{5\pi}{4}$	-.71	-.71	-1.42
$\frac{3\pi}{2}$	-1.00	.00	-1.00
$\frac{7\pi}{4}$	-.71	.71	.00
2π	.00	1.00	1.00

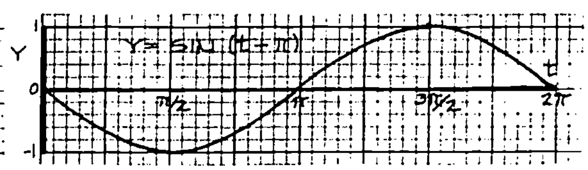
This curve has the same period as the individual waves, 2π time units. However, its amplitude is not twice as great as the amplitude of the individual waves but only approximately 1.42 times as great.

The curves $y = \sin t$ and $y = \sin (t + \frac{\pi}{2})$ are alike except that the curve represented by $y = \sin (t + \frac{\pi}{2})$ is shifted $\frac{\pi}{2}$ time units to the left. The horizontal coordinates of adjacent peaks on the two curves differ by $\frac{\pi}{2}$.

These two waves are said to be $\frac{\pi}{2}$ out of phase (or 90 degrees out of phase).

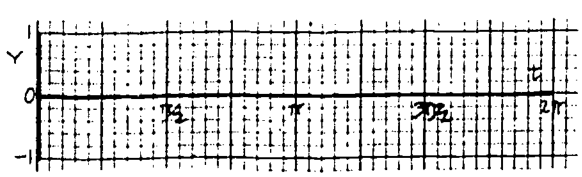
As a third case, we will consider two waves that are π (or 180 degrees) out of phase. These waves are expressed by $y = \sin t$ and $y = \sin (t + \pi)$. Notice that the horizontal distance between peaks of the two curves is π units.





The values of y corresponding to different values of t are given in the next table for each curve, and their sums are also given.

Observe that at every value of t the sum of $\sin t$ and $\sin (t + \pi)$ is zero; the two waves cancel each other, and the graph of $y = \sin t + \sin (t + \pi)$ is everywhere zero.



t	$\sin t$	$\sin (t + \pi)$	$\sin t + \sin (t + \pi)$
0	.00	.00	.00
$\frac{\pi}{4}$	.71	-.71	.00
$\frac{\pi}{2}$	1.00	-1.00	.00
$\frac{3\pi}{4}$	.71	-.71	.00
π	.00	.00	.00
$\frac{5\pi}{4}$	-.71	.71	.00
$\frac{3\pi}{2}$	-1.00	1.00	.00
$\frac{7\pi}{4}$	-.71	.71	.00
2π	.00	.00	.00

This cancellation occurs whenever two sine waves of equal amplitude and period are π radians out of phase.

You have now seen what happens when we add waves which are in phase, $\frac{\pi}{2}$ out of phase and π out of phase. We can define phase in the following way.

The <u>phase</u>, P, is the absolute value of the horizontal distance between adjacent peaks of two waves. If P = 0 the waves are said to be <u>in phase</u>. Otherwise they are said to be P units out of phase.

The phase relationship between two waves is important in the study of both sound and optics. For instance, if the sound waves originating from two speakers are π radians out of phase at a particular point, the waves can cancel each other at that point.

PROBLEM SET 26:

A set of graphs and a table appear on the following pages. We will use them to investigate what happens when two out-of-phase sine waves are added. Your teacher will assign you a value of n. You will then be asked to graph the sum of two sine waves. The two waves will be $\frac{n\pi}{8}$ out of phase with each other. Your teacher will assign 9 different values of n to 9 different groups in the class. As n increases from 0 to 8 the resulting graphs will change. All of this will become more clear when you actually do the graphing. Next time in class your instructor will describe the trends as $\frac{n\pi}{8}$ increases from 0 to π.

<u>GRAPHING METHODS</u>:

1. Use a compass to measure vertical coordinates and construct a graph of the function

$$y = \sin t + \sin \left(t + \frac{n\pi}{8}\right)$$

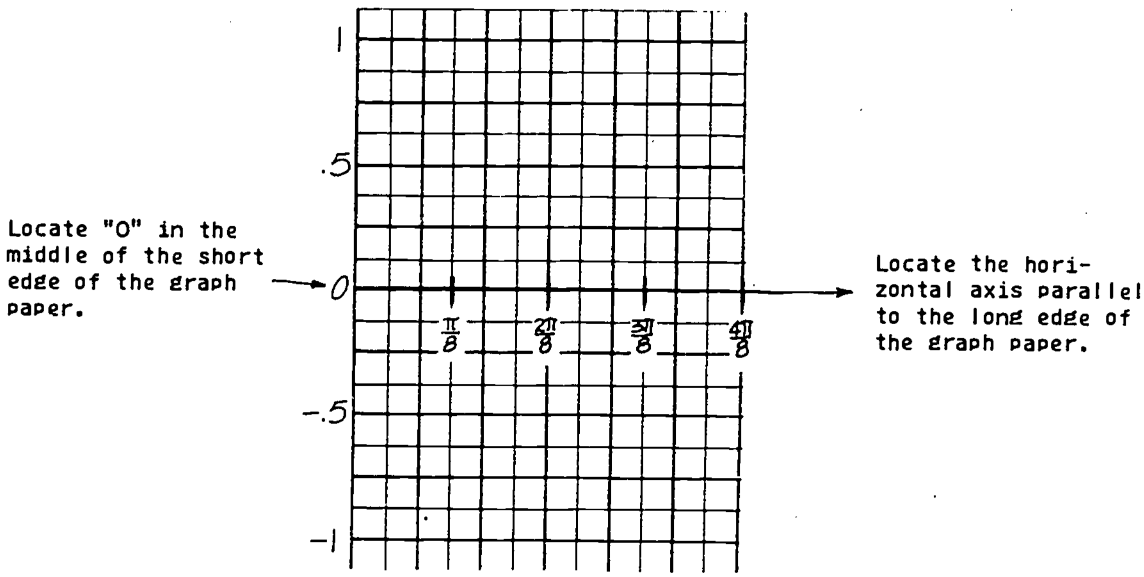
You will make your measurements on the sine curves shown on the following pages. Your instructor will give you detailed instructions on this method.

2. In this section you do not use a compass to add the vertical displacements. You do it by computation. The n = 0 row in the table has the vertical displacements corresponding to $y = \sin\left(t + \frac{(0)\pi}{8}\right)$ for t increasing from 0 to 2π in steps of $\frac{\pi}{8}$. Similarly, the nth row in the table has vertical displacements for $y = \sin\left(t + \frac{n\pi}{8}\right)$. You add the 0 row value to the nth row value to get the value corresponding to the graph of

$$y = \sin\left(t + \frac{(0)\pi}{8}\right) + \sin\left(t + \frac{n\pi}{8}\right).$$

Don't forget that you are adding <u>signed</u> numbers!

3. Use the scaling on the following page to obtain a graph having the same scale as the compass-method graphs.



Entries in the table are the values for y when $y = \sin\left(t + \frac{n\pi}{8}\right)$. The numbers are the vertical coordinates for the corresponding graphs.

n= \ t=	0	$\frac{\pi}{8}$	$\frac{2\pi}{8}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}$	$\frac{5\pi}{8}$	$\frac{6\pi}{8}$	$\frac{7\pi}{8}$	π	$\frac{9\pi}{8}$	$\frac{10\pi}{8}$	$\frac{11\pi}{8}$	$\frac{12\pi}{8}$	$\frac{13\pi}{8}$	$\frac{14\pi}{8}$	$\frac{15\pi}{8}$	2π
0	0	.38	.71	.92	1	.92	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0
1	.38	.71	.92	1	.92	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38
2	.71	.92	1	.92	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71
3	.92	1	.92	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71	.92
4	1	.92	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71	.92	1
5	.92	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71	.92	1	.92
6	.71	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71	.92	1	.92	.71
7	.38	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71	.92	1	.92	.71	.38
8	0	-.38	-.71	-.92	-1	-.92	-.71	-.38	0	.38	.71	.92	1	.92	.71	.38	0

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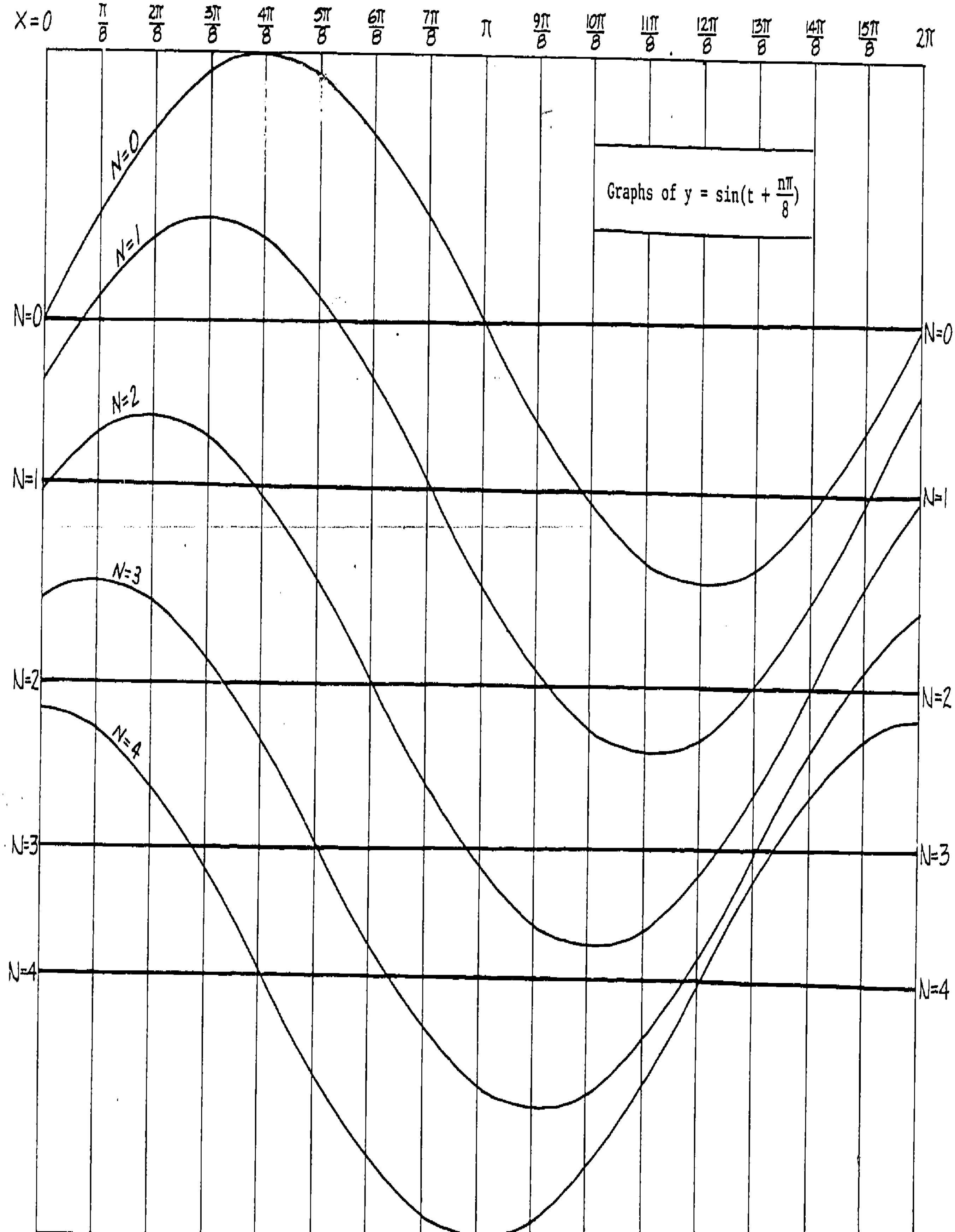
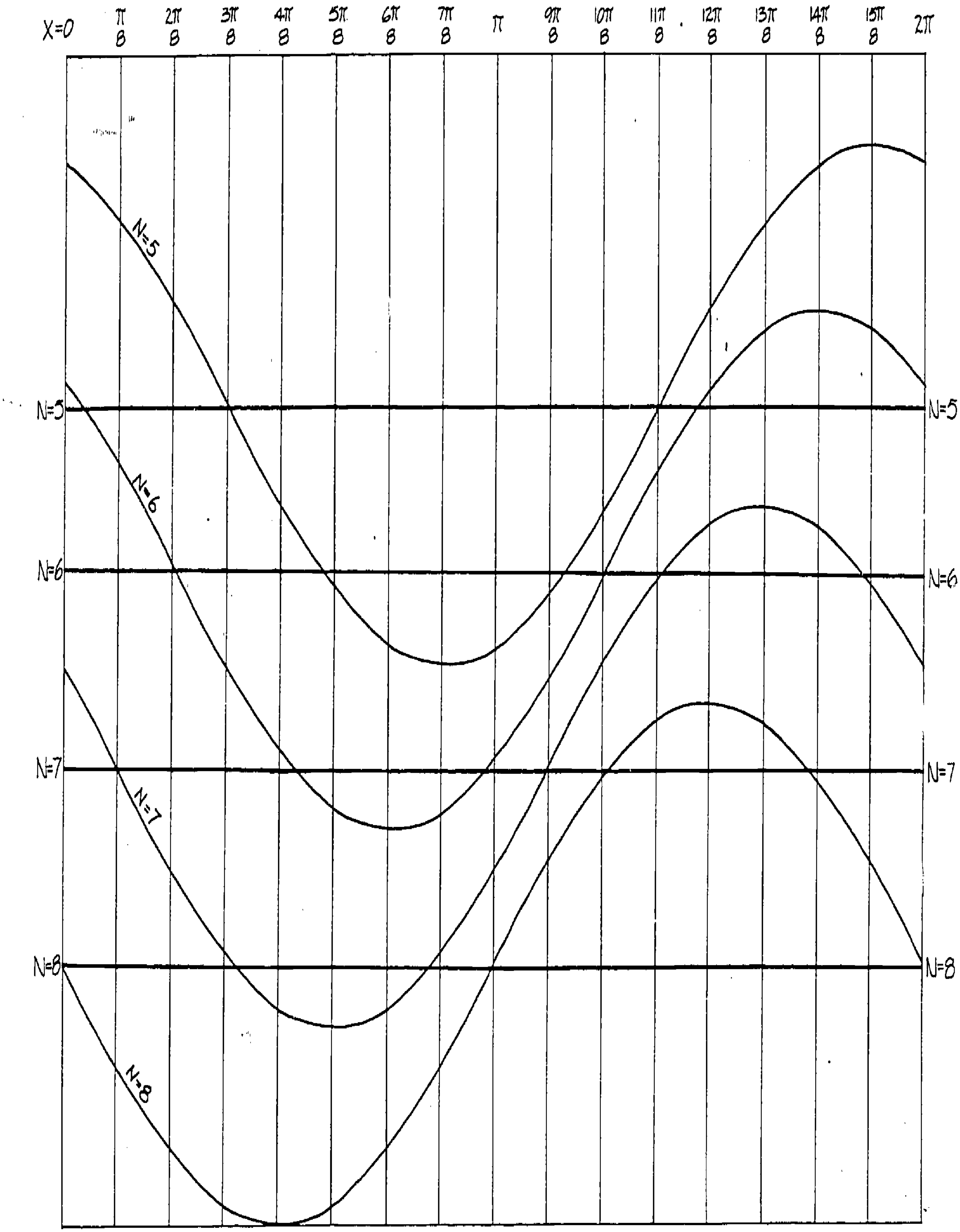


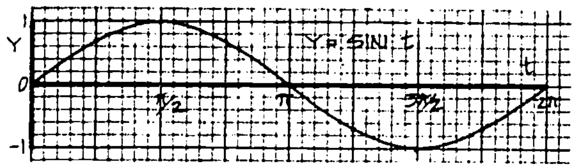
X=0
$\frac{\pi}{8}$
$\frac{2\pi}{8}$
$\frac{3\pi}{8}$
$\frac{4\pi}{8}$
$\frac{5\pi}{8}$
$\frac{6\pi}{8}$
$\frac{7\pi}{8}$
π
$\frac{9\pi}{8}$
$\frac{10\pi}{8}$
$\frac{11\pi}{8}$
$\frac{12\pi}{8}$
$\frac{13\pi}{8}$
$\frac{14\pi}{8}$
$\frac{15\pi}{8}$
2π
Graphs of $y = \sin(t + \frac{n\pi}{8})$
N=0
N=1
N=2
N=3
N=4

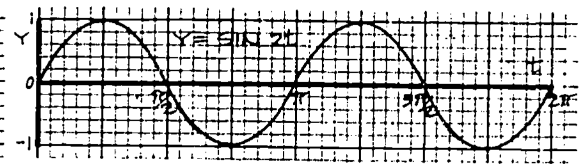

X=0
π/8
2π/8
3π/8
4π/8
5π/8
6π/8
7π/8
π
9π/8
10π/8
11π/8
12π/8
13π/8
14π/8
15π/8
2π
N=5
N=6
N=7
N=8
N=5
N=6
N=7
N=8
N=5
N=6
N=7
N=8

SECTION 27: ADDITION OF WAVES, $y = \sin mt + \sin nt$

In Section 26 we added pairs of waves that had the same amplitude and period. But musical chords are composed of tones of different periods (and frequencies). What happens when waves of different periods add together? In this section we will investigate this question. To make things simple we will not vary the amplitude; it will always be 1 for each of the sine curves to be added.

We will begin by adding sin t and sin 2t.



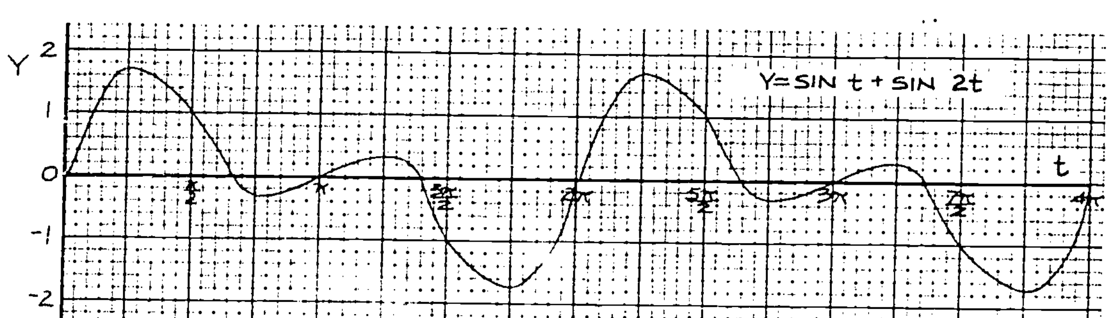


One wave has twice the period (and half the frequency) of the other. The period of sin t is 2π time units, while the period of sin 2t is π units of time.

The following table gives the values of sin t and sin 2t for various values of t and also gives the sum (sin t + sin 2t).

t	sin t	sin 2t	sin t + sin 2t
0	.00	.00	.00
$\frac{\pi}{4}$	.71	1.00	1.71
$\frac{\pi}{2}$	1.00	.00	1.00
$\frac{3\pi}{4}$	.71	-1.00	-.29
π	.00	.00	.00
$\frac{5\pi}{4}$	-.71	1.00	.29
$\frac{3\pi}{2}$	-1.00	.00	-1.00
$\frac{7\pi}{4}$	-.71	-1.00	-1.71
2π	.00	.00	.00

These values are plotted in the graph at the right. The domain of t is extended to 4π to make a point.



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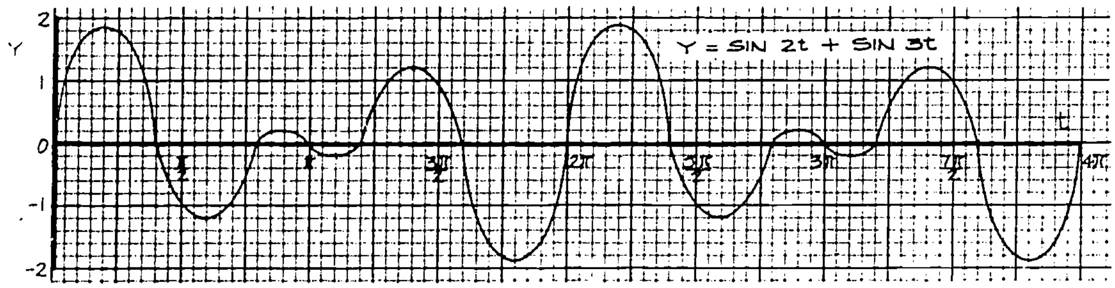
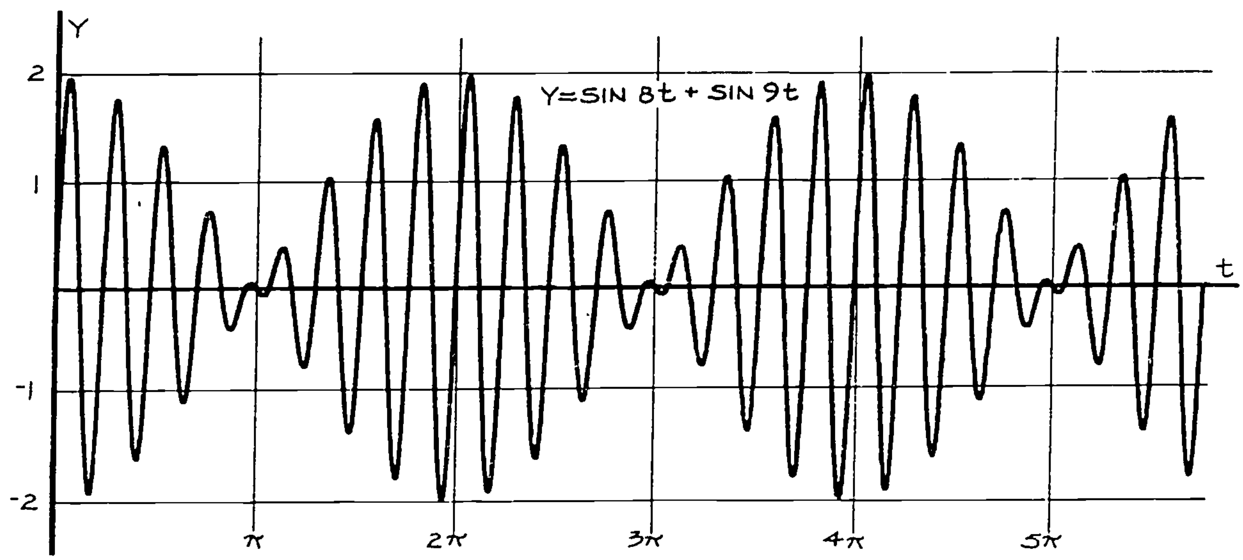
The sum of sin t and sin 2t is not a sine curve. Rather, it appears to be an alternation of two sine curves, one with greater amplitude than the other.

The shape of this curve is characteristic of the sum of any two sine waves where one has twice the period of the other, that is, the sum of sin kt and sin 2kt.

For comparison with sin t + sin 2t, the next graph shows the sum of sin 2t and sin 3t for a domain from 0 to 4π.



The pattern repeats every 2π time units, as the sum of sin t and sin 2t does. However, in the case of sin 2t + sin 3t the graph appears to be an alternation of three waves of different amplitudes. As a third example, a graph of y = sin 8t + sin 9t is shown below.



A pattern is repeated every 2π time units. The amplitude varies from almost twice the amplitude of the individual waves to nearly zero, then increases again to its maximum value.

If we plotted a graph of sin nt + sin (n + 1)t for any value of n, we would obtain a similar pattern. Over a period of 2π time units there are n + 1 wave crests; the amplitude varies from approximately twice that of the single waves to nearly zero.

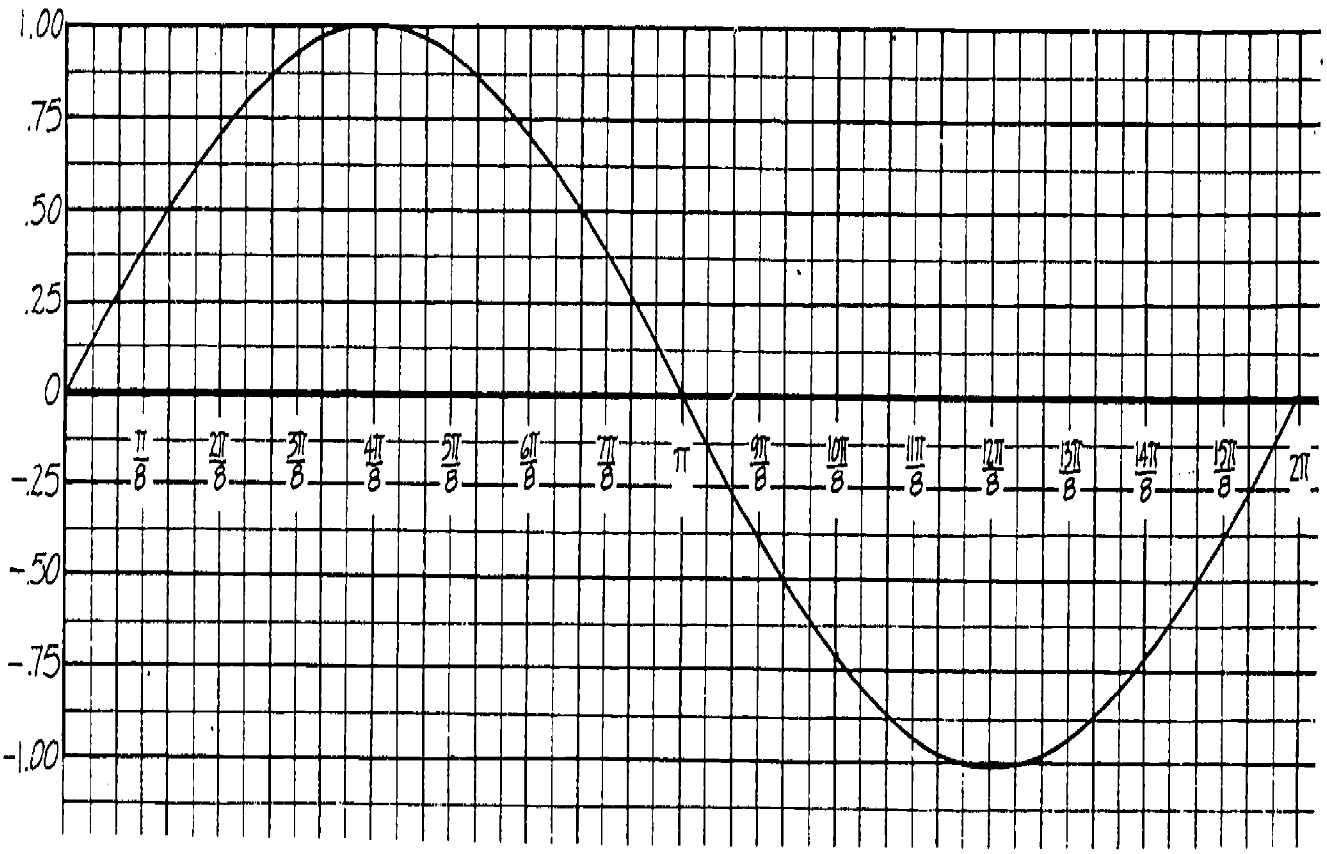
Can you imagine how a combination of two waves of nearly the same frequency sounds? We will discuss this topic in the next section.

PROBLEM SET 27:

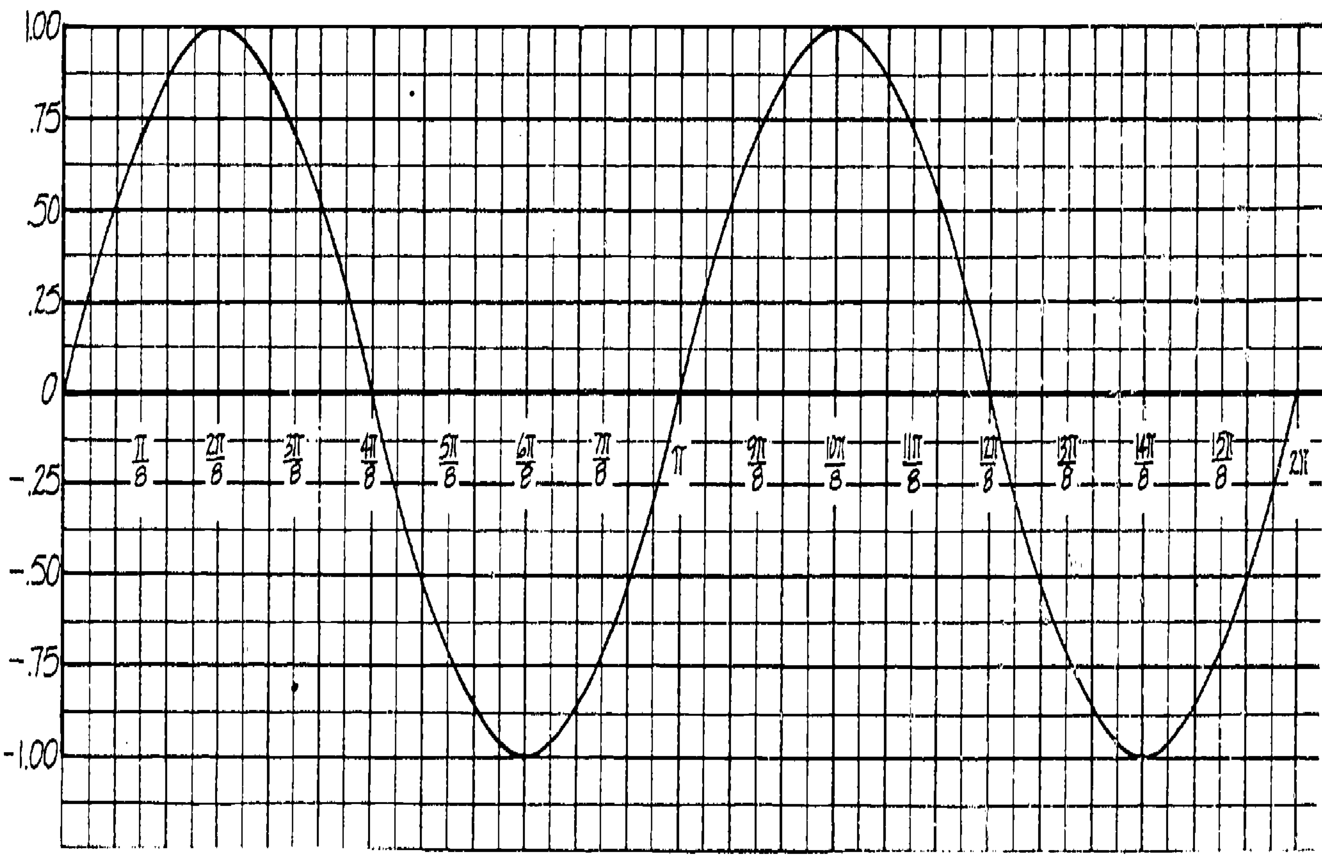
1. a. Your teacher will assign a pair of values of m and n.

   b. Use the compass method to construct a graph of y = sin mt + sin nt.

SIN t



SIN 2t



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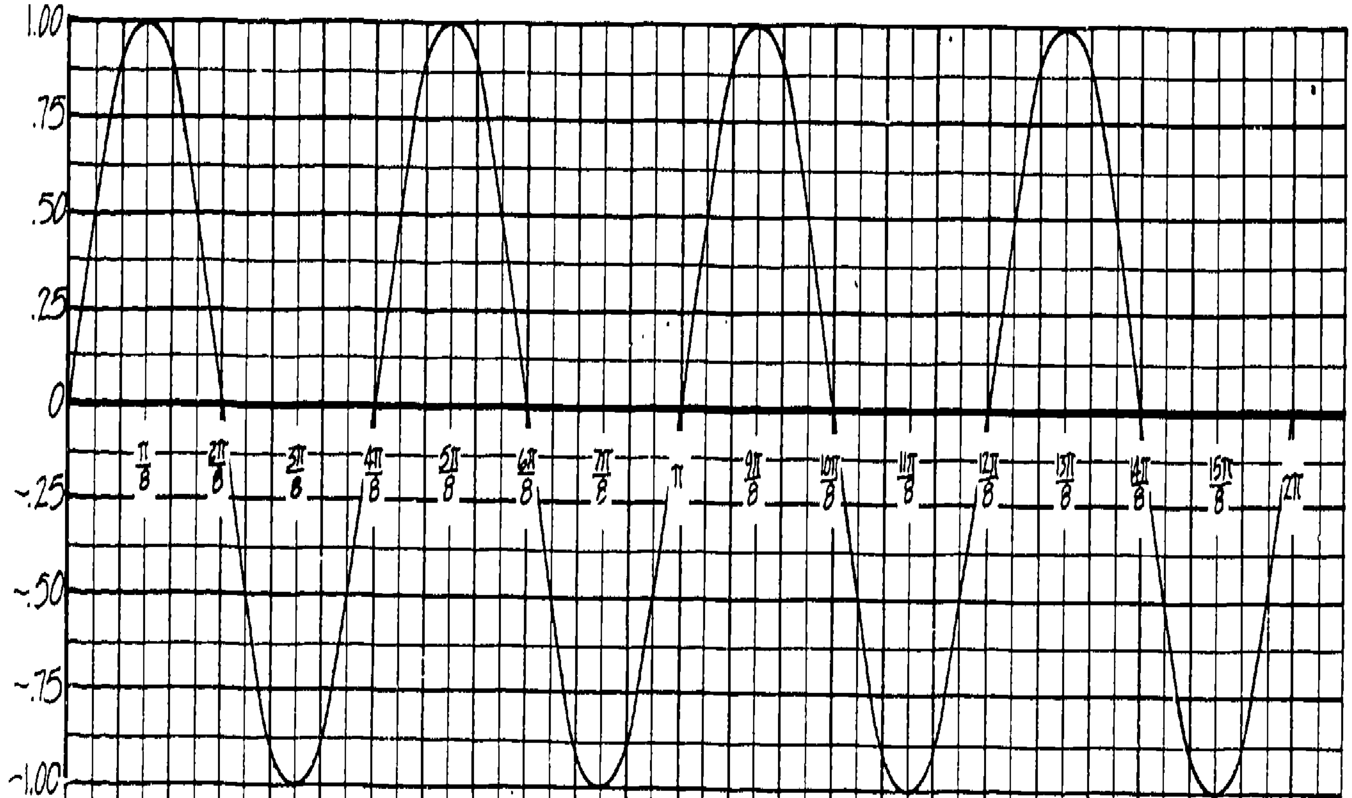
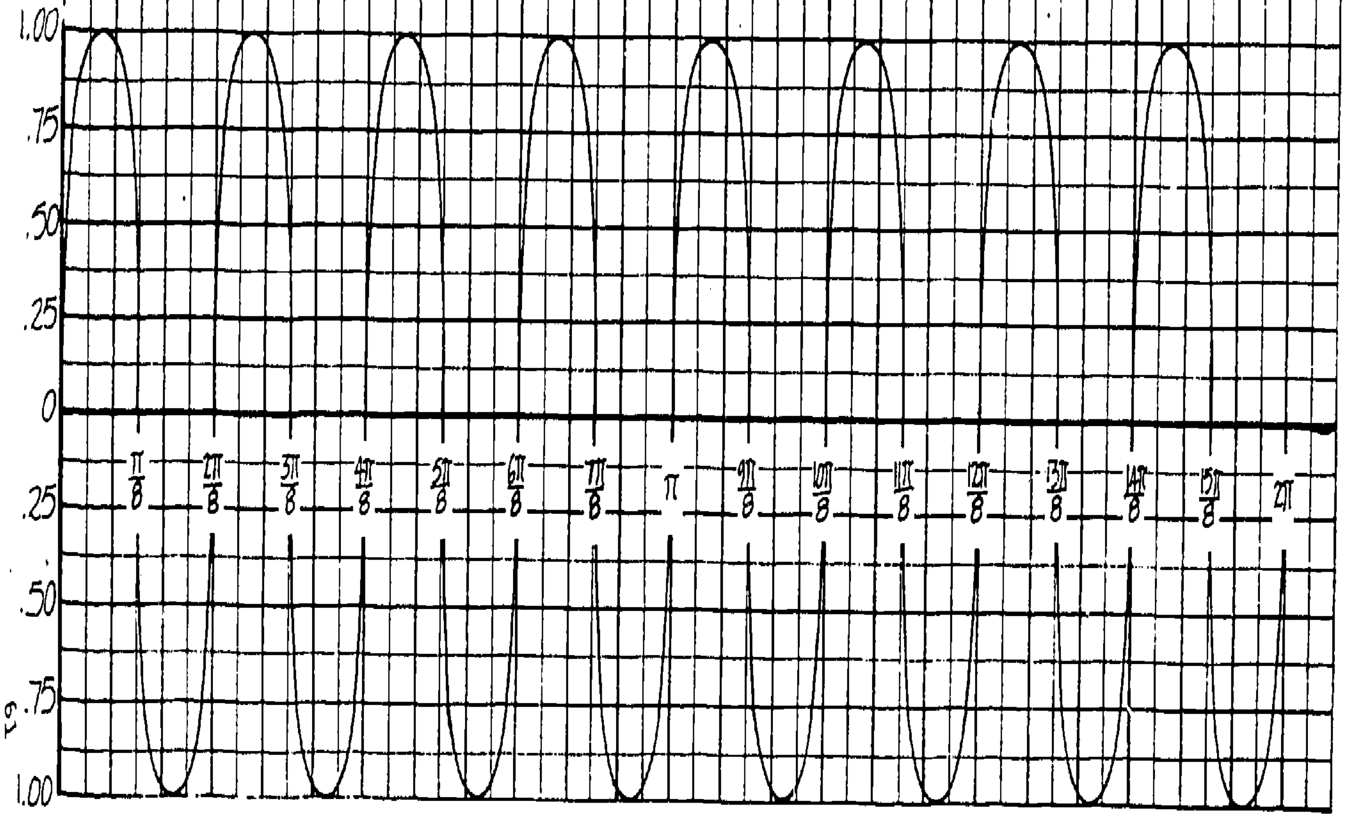
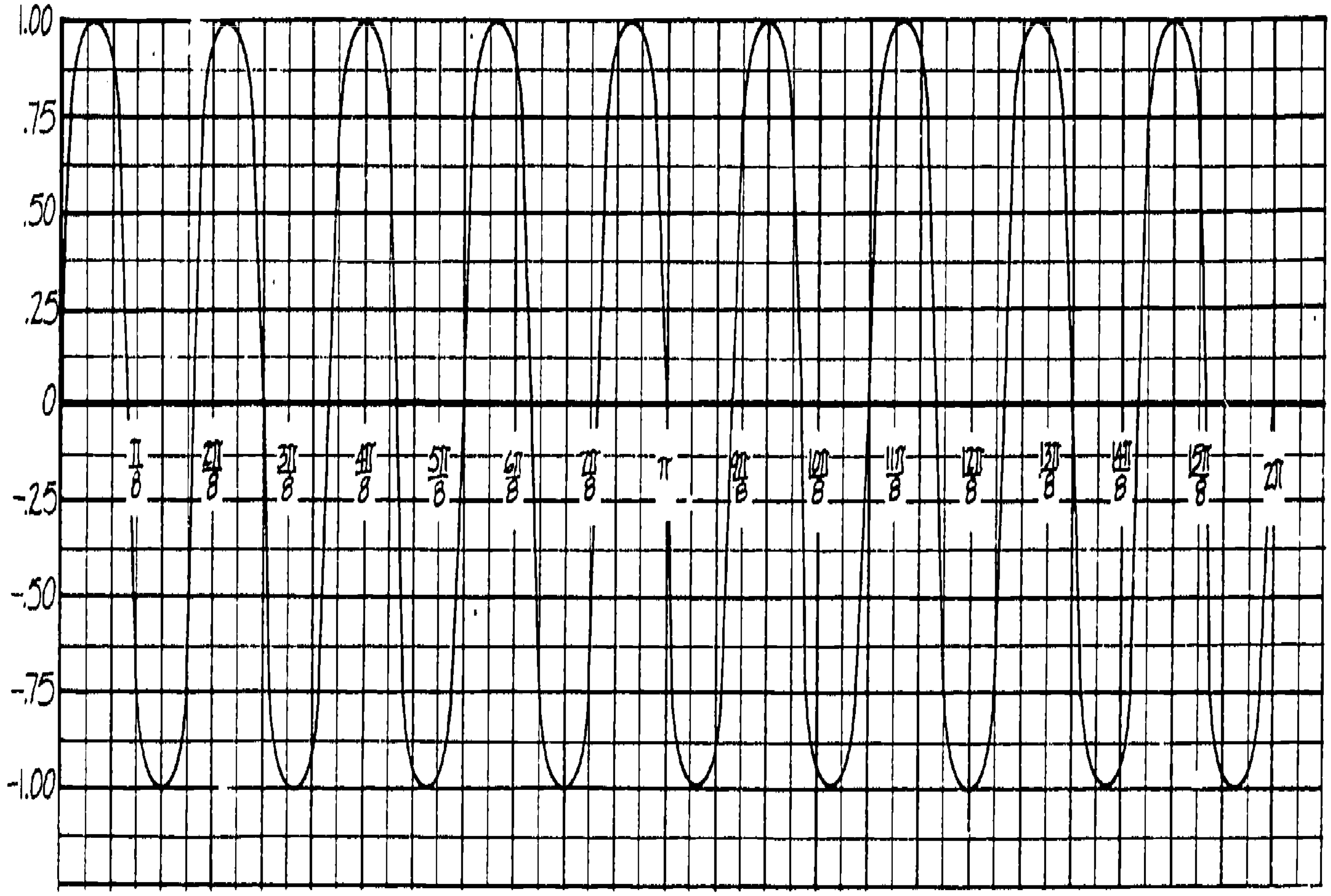
SIN 4t


1.00
.75
.50
.25
0
-.25
-.50
-.75
-1.00
π/8 2π/8 3π/8 4π/8 5π/8 6π/8 7π/8 π 9π/8 10π/8 11π/8 12π/8 13π/8 14π/8 15π/8 2π


1.00
.75
.50
.25
0
.25
.50
.75
1.00
π/8 2π/8 3π/8 4π/8 5π/8 6π/8 7π/8 π 9π/8 10π/8 11π/8 12π/8 13π/8 14π/8 15π/8 2π

SIN 8t

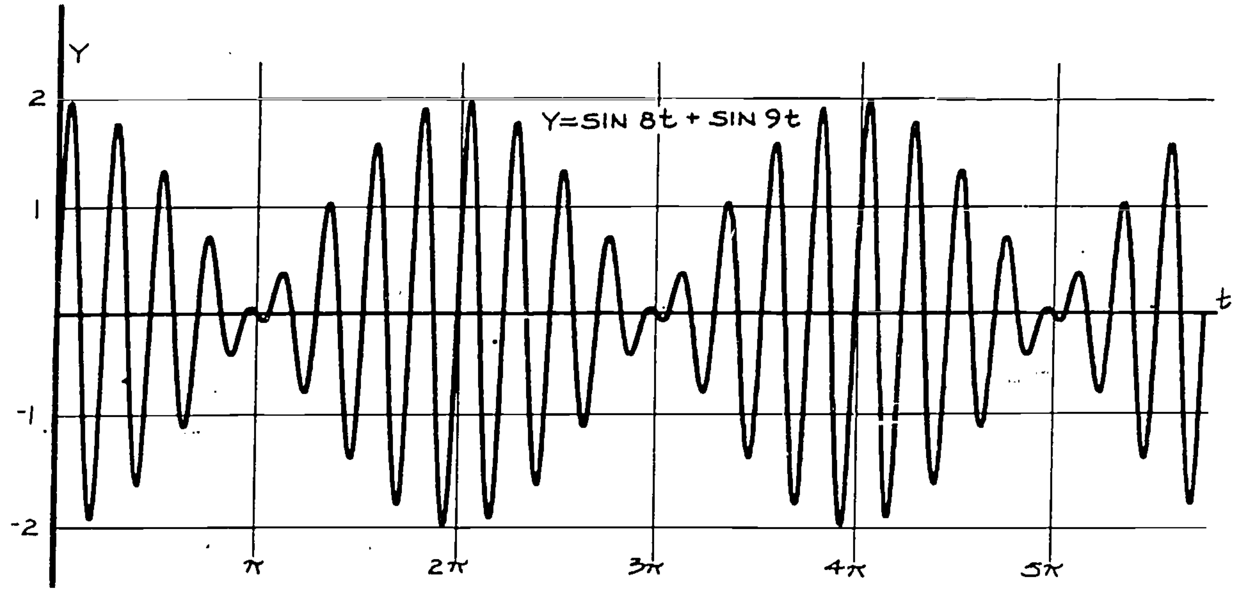
SIN 9t


1.00
.75
.50
.25
0
-.25
-.50
-.75
-1.00
π/8
2π/8
3π/8
4π/8
5π/8
6π/8
7π/8
π
9π/8
10π/8
11π/8
12π/8
13π/8
14π/8
15π/8
2π

SECTION 28: BEATS AND HARMONICS

28-1 Beats

Look again at the graph of $y = \sin 8t + \sin 9t$.



This graph represents the combination of two waves with slightly different frequencies. The frequency of sin 8t (see Section 24-2) is

$$f = \frac{8}{2\pi}$$

$$= \frac{4}{\pi} \frac{\text{cycles}}{\text{unit time}}$$

The frequency of sin 9t is

$$f = \frac{9}{2\pi}$$

$$= \frac{4.5}{\pi} \frac{\text{cycles}}{\text{unit time}}$$

The amplitude of the sum of these two waves varies in a regular fashion. Over a period of 2π time units the amplitude decreases from its maximum value to almost zero, then increases again to a maximum.

The loudness of a sound is related to amplitude. A sound produced by combining two tones of almost, but not quite, the same frequency has varying loudness. The combination of waves expressed by $y = \sin 8t$ plus $y = \sin 9t$ is loud, then soft, then loud again over an interval of 2π time units. We hear this variation in loudness as beats. You will have an opportunity to hear beats in Science class. You may have difficulty at first recognizing what you are hearing. Beats are used by musicians to tune instruments.

The frequency of beats is equal to the difference in frequencies of the two waves that are being added. For example, if t is measured in seconds the combination of sin 8t and sin 9t produces a beat every 2π seconds, so the beat frequency is $\frac{1}{2\pi}$ beats per second. The difference in frequency between sin 8t and sin 9t is

$$\frac{4.5}{\pi} - \frac{4}{\pi} = \frac{.5}{\pi}$$

$$= \frac{1}{2\pi} \text{ cycles per second}$$

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The closer the frequencies of two tones are, the slower is the frequency of beats. The more different the frequencies of two tones, the more frequent the beats are.

EXAMPLE:

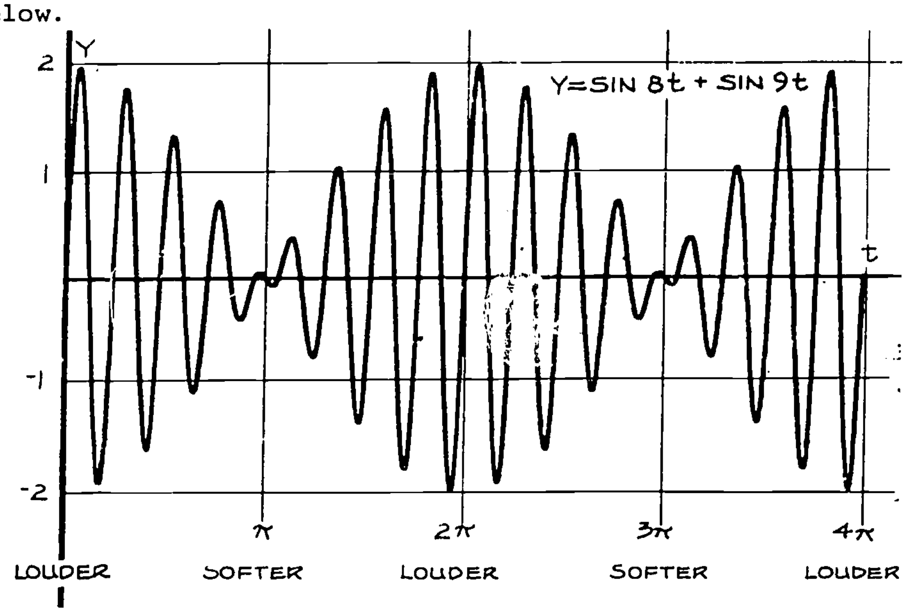
The frequencies of two tones are 256 cycles per second and 258 cycles per second. If the two tones are combined, how many beats are heard per second?

SOLUTION:

The beat frequency is the difference between the frequencies of the two tones.

$$258 - 256 = 2\ \frac{\text{beats}}{\text{sec}}$$

Beats may also be observed with an oscilloscope. If an oscilloscope were displaying y = sin 8t + sin 9t it might show a much smaller portion of the curve than in the graph below.



You might think of the oscilloscope screen as a window through which you watch the wave as it passes by. As the beat occurs you will see the parts of the curve labeled "louder." Between beats, the amplitude of the curve decreases, and you will see the parts labeled "softer."

28-2 Harmonics

Middle C has a frequency of 256 cycles per second (cps). High C has a frequency of 512 cps, exactly double the frequency of middle C. When the two notes are played together, the resulting sound is pleasing to the ear, harmonious. In fact, high C is called a harmonic of middle C. There are many other harmonics of middle C as well. Any frequency that is a whole number multiple of middle C is a harmonic of middle C. When middle C is mixed with any harmonic the resulting sound will be pleasing to the ear.

Because each frequency has more than one harmonic, each harmonic has a number. The first harmonic frequency which is higher than the original frequency is called the "first harmonic." High C with a frequency of 512 cps is the first harmonic of middle C. On the following page we list the frequencies of each succeeding harmonic to middle C.

Frequency (cps)	Pitch	Name
256	Middle C	Fundamental Frequency
2 x 256 = 512	High C (one octabe above middle C)	First Harmonic
3 x 256 = 768	G, one octave above middle G	Second Harmonic
4 x 256 = 1024	C, two octaves above middle C	Third Harmonic
5 x 256 = 1280	E, two octaves above middle E	Fourth Harmonic
.		.
.		.
.		.
n x 256 ↑ (whole number)		(n-1)th Harmonic

PROBLEM SET 28:

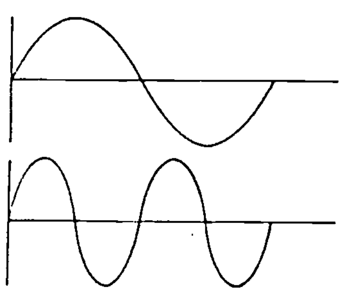
For Problems 1 through 7, state the beat frequencies of the given pairs of frequencies. (1 kilocycle = 1000 cycles)

1. $f_1 = 60$ cps $\qquad f_2 = 63$ cps
2. $f_1 = 543$ cps $\qquad f_2 = 543.34$ cps
3. $f_1 = 1.0003 \times 10^4$ cps $\qquad f_2 = 1.0000 \times 10^4$ cps
4. $f_1 = 3 \dfrac{\text{kilocycles}}{\text{sec}}$ $\qquad f_2 = 3.01 \dfrac{\text{kilocycles}}{\text{sec}}$
5. $f_1 = 620.035 \dfrac{\text{cycles}}{\text{sec}}$ $\qquad f_2 = 620.01$ cps
6. $f_1 = 33$ cps $\qquad f_2 = 40$ cps
7. $f_1 = 1930$ cps $\qquad f_2 = 1928$ cps
8. Add the two curves in a, b and c below and match with the correct answers, I, II and III, on the right.

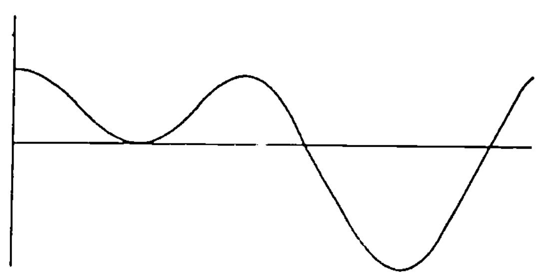
a.

$y = \sin t$

$y = \sin 2t$



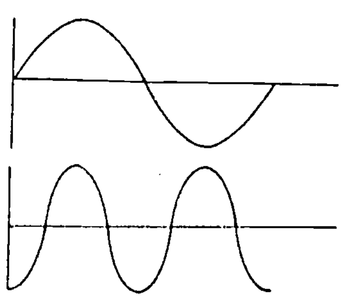
I



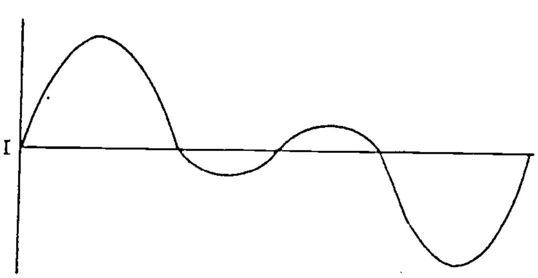
b.

$y = \sin t$

$y = \sin 2(t-\frac{\pi}{4})$



II



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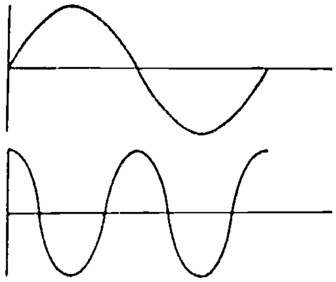
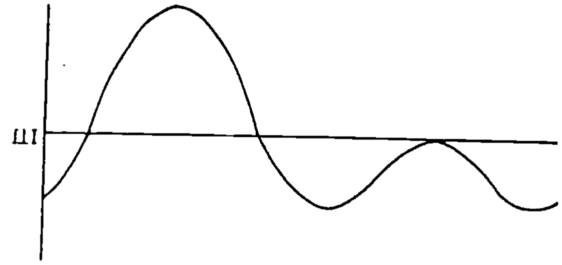
c.

y = sin t



$y = \sin 2(t + \frac{\pi}{4})$



9. Below is a set of functions of the form $y = \sin nt + \sin (n + 1)t$.

(I) $y = \sin t + \sin 2t$

(II) $y = \sin 100t + \sin 101t$

(III) $y = \sin 2t + \sin 3t$

(IV) $y = \sin 200t + \sin 201\, t$

(V) $y = \sin 8t + \sin 9t$

Rereading page 143 of Section 27 will help you in answering the following questions.

a. Which function will have the fewest peaks in the interval $0 \leq t \leq 2\pi$?

b. Which function will have the most peaks in the interval $0 \leq t \leq 2\pi$?

c. How many peaks will function III have in the interval $0 \leq t \leq 2\pi$?

d. Order the functions. Put the function with the fewest peaks per cycle first. The one with the most peaks per cycle should be last.

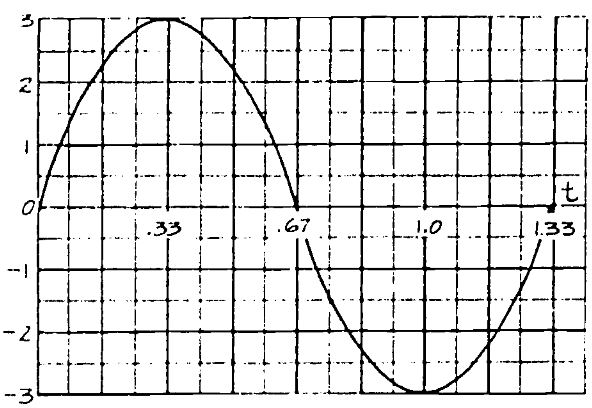
e. Which function will have the least difference in amplitude between successive peaks?

f. What is the beat frequency for each of the functions?

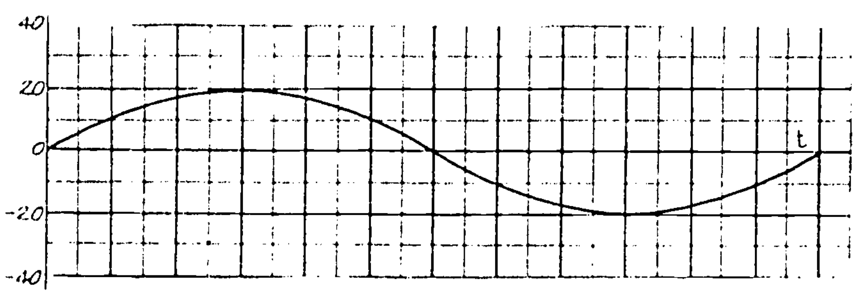
REVIEW PROBLEM SET 29:

State the amplitude of each of the sine curves below.

1.



2.

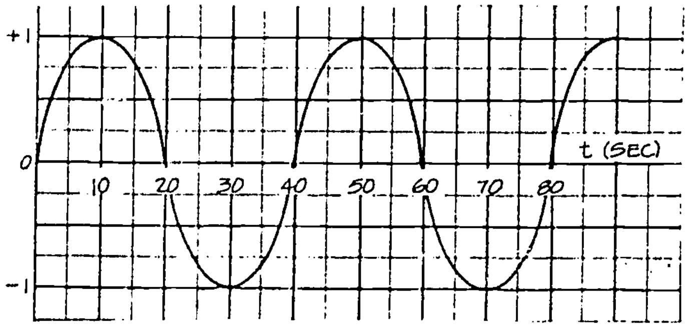


3. $y = 47 \sin \frac{3\pi}{19} t$

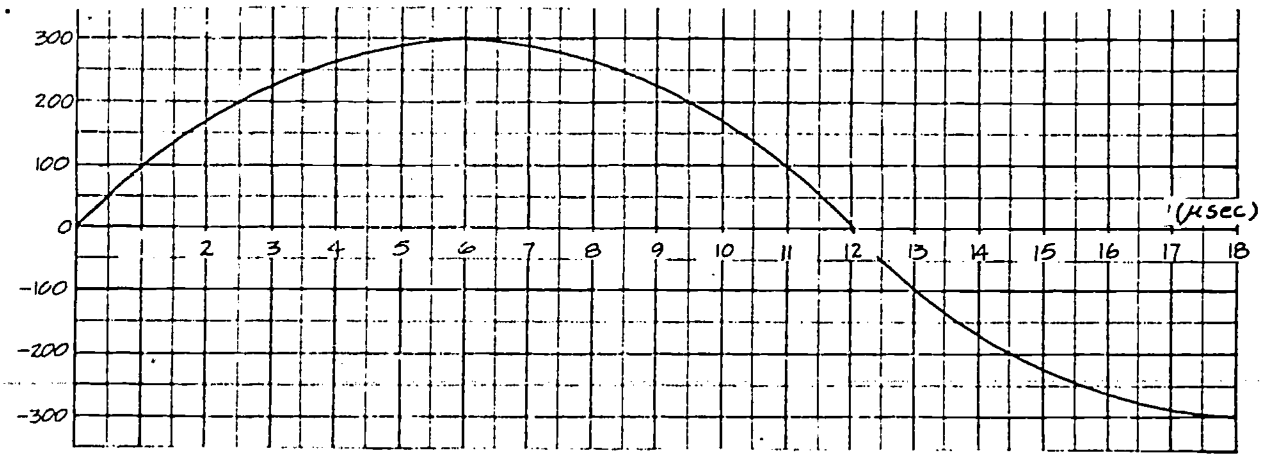
4. $y = 10^3 \sin \frac{3}{2} \pi x$

State the frequency and period of each of the sine curves below.

5.



6.



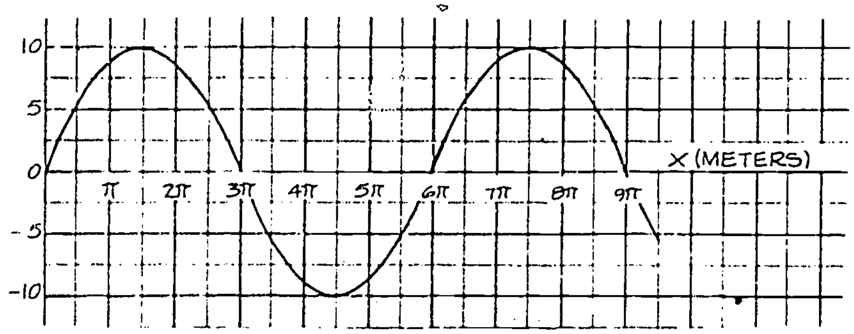
7. $y = 47 \sin bt$, t in years

8. $y = 14 \sin 2\pi t$, t in minutes

9. $y = .04 \sin 113t$, t in milliseconds

State the wavelength, λ, for each of the sine curves below and on the following page.

10.



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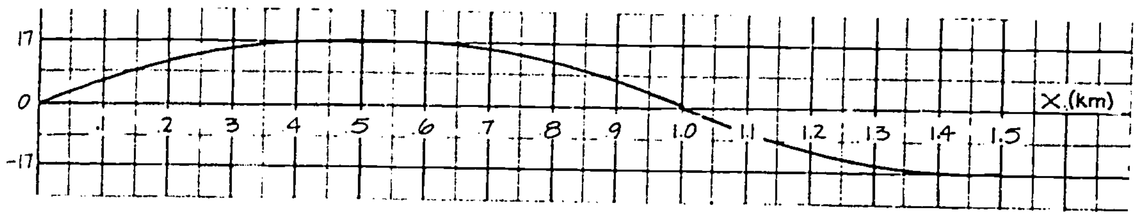
11.



12. $y = \sin cx$, x in meters

13. $y = 10^3 \sin 33x$, x in km

14. $y = \pi \sin 14\pi x$, x in mm

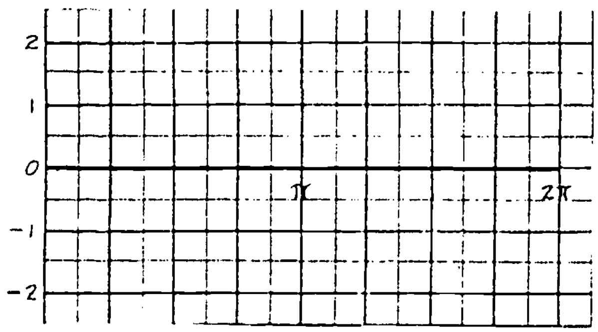
15. Below is a set of graphs that resulted from the addition of sin x and $\sin(x + \frac{n\pi}{4})$.

 a. For which graph does n = 0?

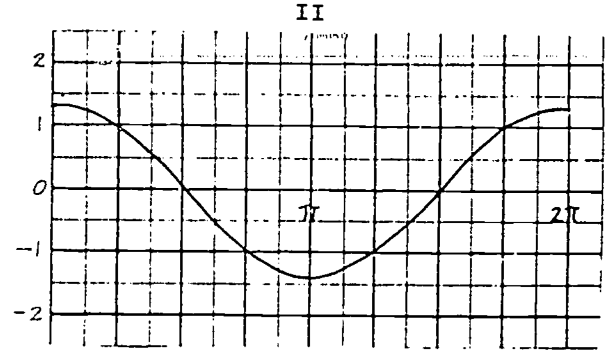
 b. For which graph does n = 4?

 c. Sequence the graphs from completely in phase to completely out of phase.

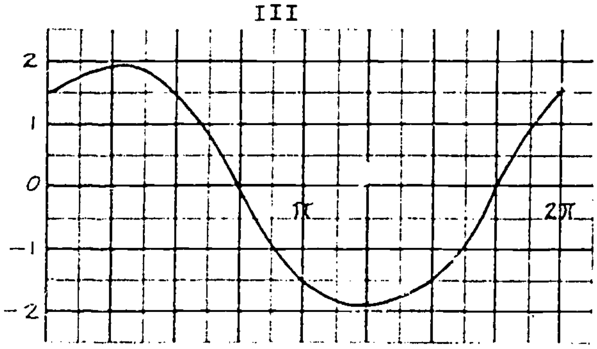
I



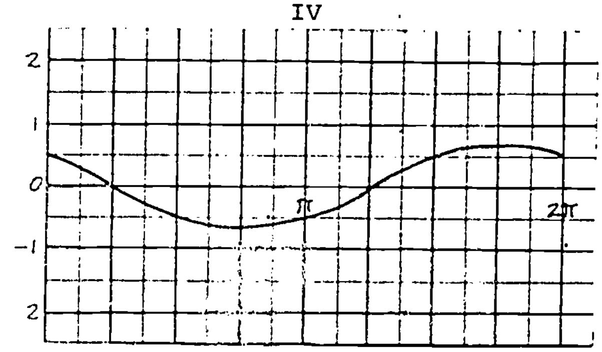
II



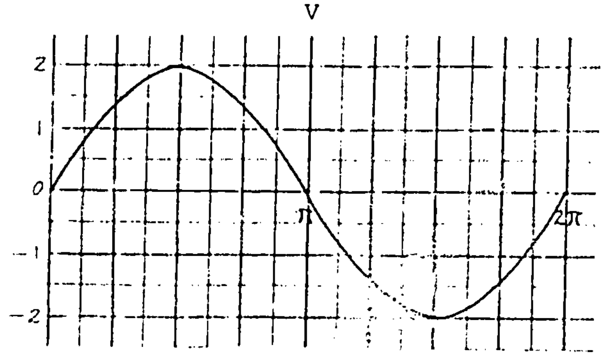
III



IV



V



16. State the beat frequencies of the following pairs of frequencies.
    a. 100 cps, 101 cps
    b. 3547 cps, 3544 cps
    c. 60 cps, 63 cps
17. Which of the following frequencies are harmonics of 500 cps?
    a. 250 cps
    b. 750 cps
    c. 100 cps
    d. 1000 cps
    e. 1500 cps

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SUPPLEMENTARY SECTIONS

SECTION X1: ABSOLUTE VALUE AND SUMMATION NOTATION

X1-1 Introduction to the Vision Lessons

The next few Mathematics and Science Lessons treat the subject of vision. This sequence is intended to help you understand how the human eye works and some of the health problems involving vision. The eye contains a lens which focuses images of the outside world. This occurs through a process called refraction in which light is bent as it enters the lens. When the eye is focused, the light is bent in just the right way so that a clear image of the outside object is reproduced on the back of the eyeball. This image is then transmitted by neurons to the brain.

In order to understand these processes you will be introduced to the subject of optics. Optics involves the study of the behavior of light. You will work with some of the basic formulas of optics that describe the refraction of light by lenses.

You will also expand your knowledge of statistics. You were introduced to one statistical measure when you studied chi square last year. In this sequence a statistical measure called the standard deviation will be discussed. In Science class you will be making a large number of measurements of the same quantity. You will see how the standard deviation can be used to make a "best estimate" of the true value of the quantity being measured.

In this section you will review absolute value and summation notation. You will use these concepts frequently as you analyze data from your Science Laboratory activities.

X1-2 Absolute Value

You have worked with absolute values several times already. For example, the amplitude of a sine curve was defined as the absolute value of the y-coordinate difference between a peak and a valley of the curve.

The absolute value of an expression is merely the non-negative value of the expression. We designate absolute value by enclosing the expression between two vertical lines. For example, the absolute value of -8 is written as $|-8|$. Its value is positive 8. We write $|-8| = 8$.

What is the value of $|3 - 7|$? This problem is easy if you carry out the calculation inside the absolute value signs first. The value of $3 - 7$ is -4. Therefore,

$$\begin{aligned} |3 - 7| &= |-4| \\ &= 4 \end{aligned}$$

XI-3 Summation Notation

When you studied vectors in Unit II you were introduced to summation notation. This notation is a shorthand for writing sums. For example, the sum

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

can be written as follows using summation notation.

$$\sum_{i=1}^{7} x_i$$

$\sum$ is the Greek capital letter sigma. In summation notation, $\sum$ stands for "sum." The expression above is read "The sum of x sub i for i = 1 to 7."

The following examples will help you interpret summation notation.

EXAMPLE 1:

Write out the sum given by $\sum_{i=3}^{5} y_i$

SOLUTION:

This expression is read "The sum of y sub i for i = 3 to 5." In order to write out the sum we substitute the numbers 3, 4, and 5 for i in the expression y_i. We obtain y_3, y_4 and y_5. Then we add these quantities to obtain

$$y_3 + y_4 + y_5$$

EXAMPLE 2:

Suppose $x_1 = 3$, $x_2 = 7$, $x_3 = 20$ and $x = 10$. Compute the numerical value of $\sum_{i=1}^{3} |x_i - x|$.

SOLUTION:

The summation expression is read "The sum of the absolute value of the quantity x sub i minus x for i = 1 to 3." Therefore we substitute the values 1, 2, and 3 for i in the expression $|x_i - x|$. We obtain $|x_1 - x|$, $|x_2 - x|$ and $|x_3 - x|$. We now use the values of x, x_1, x_2 and x_3 to simplify these expressions.

$$\begin{aligned} \sum_{i=1}^{3} |x_i - x| &= |x_1 - x| + |x_2 - x| + |x_3 - x| \\ &= |3 - 10| + |7 - 10| + |20 - 10| \\ &= |-7| + |-3| + |10| \\ &= 7 + 3 + 10 \\ &= 20 \end{aligned}$$

PROBLEM SET X1.

Find the following absolute values.

1. $|8|$ 3. $|0|$ 5. $|7 - 1|$

2. $|-6|$ 4. $|-1.34|$ 6. $|11 - 18|$

7. Suppose $x_1 = 5$, $x_2 = 6$, $x_3 = 7$ and $x_4 = 8$.

   Find $\sum_{i=1}^{4} x_i$.

In Problems 8 through 12 let $x = 8$, $x_1 = 3$, $x_2 = 6$ and $x_3 = 13$.

8. $|x_1 - x| =$

9. $|x_2 - x| =$

10. $|x_3 - x| =$

11. $\sum_{i=1}^{3} |x_i - x| =$

12. $\dfrac{\sum_{i=1}^{3} |x_i - x|}{3} =$

13. Suppose that $x_1 = 10$ and $x_2 = 4$. Compute $\sum_{i=1}^{2} |x_i - x|$ for the following values of x.

   a. $x = 10$ b. $x = 7$ c. $x = 4$

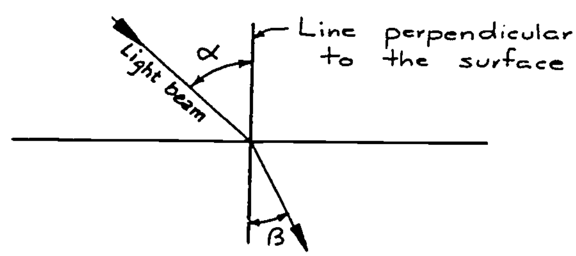
14. Again suppose that $x_1 = 10$ and $x_2 = 4$. This time compute $\sum_{i=1}^{2} (x_i - x)^2$ for the following values of x.

   a. $x = 10$ b. $x = 7$ c. $x = 4$

SECTION X2: SNELL'S LAW

X2-1 Calculations of the Index of Refraction

In Science Lesson 27 you were introduced to Snell's Law. This law governs the phenomenon of refraction. It says that when light enters a given medium from air, the ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant. This constant is called the index of refraction.



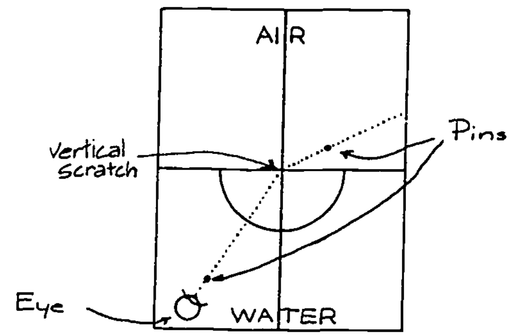
Snell's Law says

$$\text{Index of refraction} = \frac{\text{sine of angle of incidence}}{\text{sine of angle of refraction}}$$

or

$$n = \frac{\sin \alpha}{\sin \beta}$$

In Laboratory Activity 28 you observed refraction directly. You filled a semicircular plastic box with a fluid and then placed the box on a piece of graph paper as shown on the next page. By lining up the two pins and the vertical scratch on the box you produced a set of angles. You produced three graphs based on your measurements in the Laboratory Activity. Specifically, you produced one for air, one for water and one for glycerine. You will be concerned mainly with the graphs for water and glycerine.

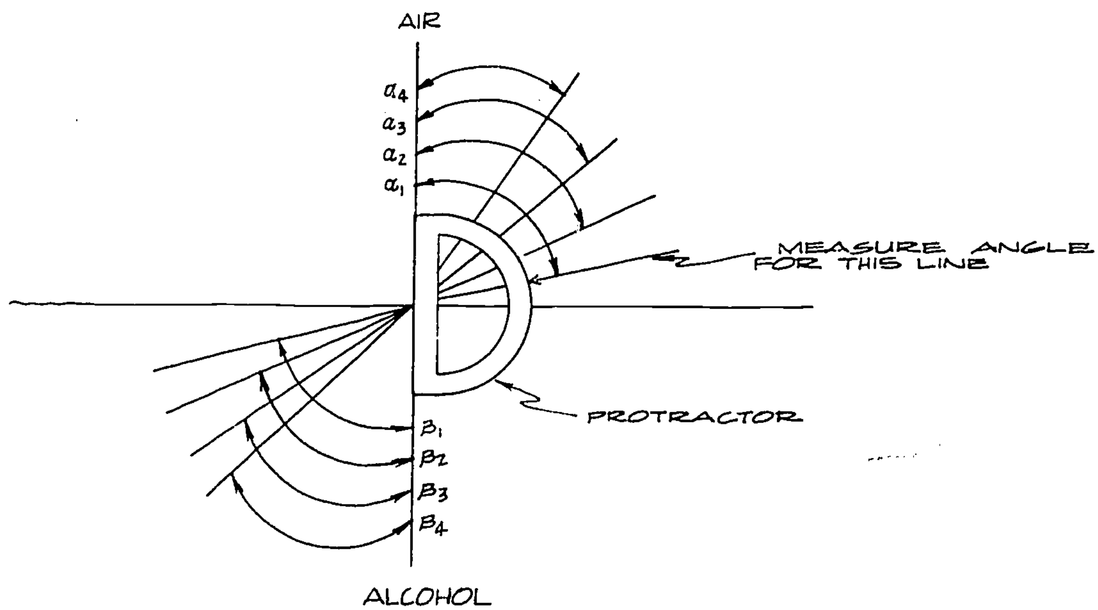


The graph on the following page represents the work of a Biomed student by the name of Irving Ball. It is clear that Irving has labeled his graph properly. He has his alphas and betas paired correctly. He has the labels "AIR" and "ALCOHOL" in the right places.

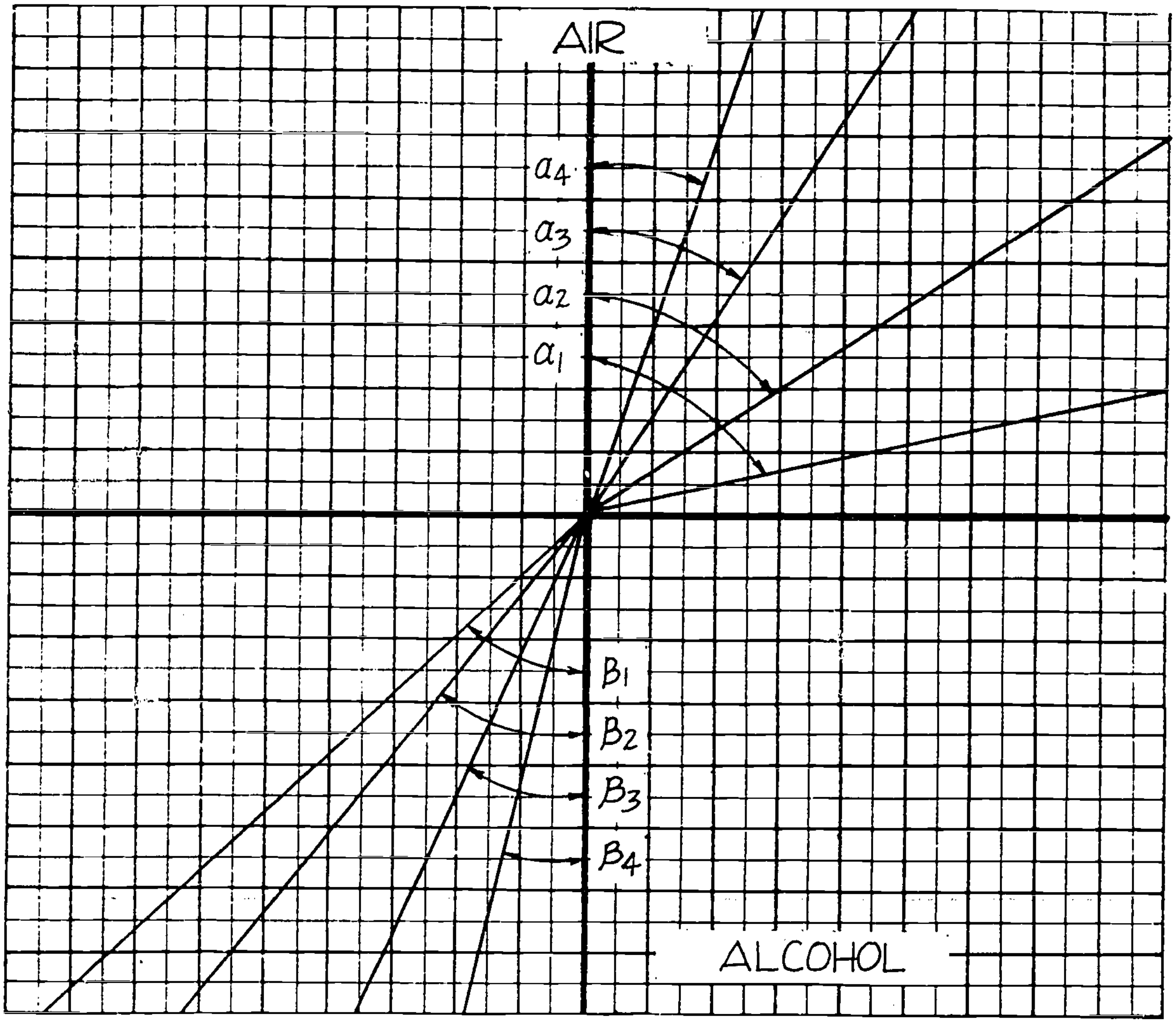
In Mathematics class he will measure the angles with a protractor. The degree measures of the angles will then be entered in Mathematics Data Sheet X2. His teacher will give him this data sheet during class. After he has entered the degree measures of the angles, he will calculate the quotients $\frac{\sin \alpha_i}{\sin \beta_i}$ in order to determine the index of refraction. We will follow Irving's progress as he completes his work.

Irving receives two copies of Math Data Sheet X2. He writes "alcohol" in the blank at the top of one data sheet. You will not write "alcohol" because your measurements were performed with water and glycerine. You will enter "water" on one data sheet and "glycerine" on the other.

He takes his protractor and measures α_1.



He finds that $\alpha_1 = 77°$. (You may wish to check his work by measuring α_1 on the sample graph. This will also refresh your memory on the use of a protractor.) He then enters 77° in the first row of the column under "α" on the data sheet. In a similar manner he measures and records the rest of the α's. He then measures the β's. He finds that $\beta_1 = 46°$. (You should check his work by measuring β_1 on



the sample graph. This exercise will ensure that you will be able to find β_1 on your own graph.) He enters 46° in the first row of the β_1 column on the data sheet. In a similar fashion he measures the other β's and enters them on the data sheet. He has now completed measuring all of his angles. Part of the first row of his data sheet is shown below.

NAME: I. Ball

Trial Number	α	β	sin α	sin β	n	
1	77°	46°				

Irving now fills in the next three boxes in the first row of his table. He consults the Trig Table at the end of this book to obtain the following information.

$$\sin 77° \approx .974$$

$$\sin 46° \approx .719$$

He now uses the formula

$$n = \frac{\sin \alpha}{\sin \beta}$$

for Snell's law to calculate n_1. If Irving had a slide rule or calculator he would use it to calculate the quotient (.974 ÷ .719). Since he doesn't he is forced to use long division.

```
        1.35
719 / 974.00
      719
      2550
      2157
       3930
       3595
        335
```

Irving's teacher has told him to calculate n_1 to the nearest hundredth. He notices that the next digit in his quotient is less than 5, because 335 is less than half of 719. He therefore reports n_1 as

$$n_1 \approx 1.35$$

The first row of Irving's data sheet now looks like this.

Trial Number	α	β	sin α	sin β	n	
1	77°	46°	.974	.719	1.35	

You should perform these calculations for your water and glycerine data and record your results as Irving did.

X2-2 Applications of Snell's Law

Snell's Law is not just something designed to give Biomed students headaches. Scientists have found many uses for it. One use is in determining the concentrations of solutions. For example, if the index of refraction of a sugar solution is known, then the concentration may be determined. Scientists have constructed detailed tables that relate the concentration of a sucrose solution to its index of refraction.

Another use of Snell's Law is to calculate the speed at which light travels through a substance. This relation was developed in the Science course.

$$n_{\text{substance x}} = \frac{c_{\text{air}}}{c_{\text{substance x}}}$$

where $n_{\text{substance x}}$ = index of refraction of substance x

c_{air} = speed of light in air

$c_{\text{substance x}}$ = speed of light in substance x

Consider the example on the next page. We know that $c_{\text{air}} \approx 3 \times 10^8 \frac{\text{m}}{\text{sec}}$ and also that the index of refraction of substance x is 1.2. We can find the speed of light in substance x by substituting into the formula.

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$$n_x = \frac{c_{air}}{c_x}$$

$$1.2 \approx \frac{3 \times 10^8}{c_x}$$

$$c_x \approx \frac{3 \times 10^8}{1.2}$$

$$c_x \approx 2.5 \times 10^8 \frac{m}{sec}$$

PROBLEM SET X2:

1. A light beam entered a medium from air. The angle of incidence was 45°, and the angle of refraction was 27°. What was the medium? The index of refraction (with respect to air) for methyl alcohol is 1.33, for ethyl alcohol is 1.36, and for glass is 1.56.

2. One way to determine the concentration of a sugar solution is to find the index of refraction of the solution. Suppose a sugar solution refracts light as follows.

angle of incidence = 58°

angle of refraction = 35°

Determine the concentration of the solution by calculating the index of refraction and referring to the table at the right.

Concentration of Sucrose in $\frac{\text{grams sucrose}}{\text{100 grams solution}}$	Index of Refraction
0	1.33
25	1.37
50	1.42
75	1.48

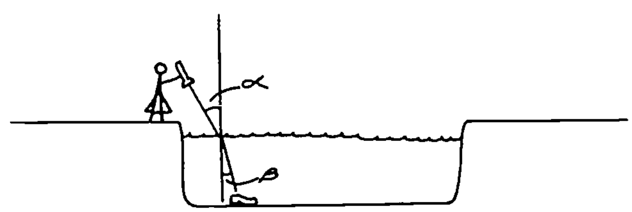
3. The speed of light in carbon disulfide is $1.84 \times 10^8 \frac{m}{sec}$. Find the index of refraction of carbon disulfide with respect to air ($c_{air} = 3.00 \times 10^8 \frac{m}{sec}$).

4. The index of refraction of quartz with respect to air is n = 1.54. What is the speed of light in quartz?

5. At an evening party celebrating the completion of finals, Keith threw sassy Sally's shoe into the swimming pool, which had been filled with quinine water. Consequently, Sally found a flashlight and aimed it into the pool to find her shoe.

a. Which is the angle of incidence (α or β)? the angle of refraction?

b. If the angle of refraction is 32°, when Sally sees her shoe, what is the angle of incidence? ($n_{quinine\ water} \approx 1.37$)



c. What is the speed of light streaking through the quinine water toward the shoe?

SECTION X3: THE MEAN AND THE MEAN DEVIATION

X3-1 Why Did We Make Several Measurements of the Same Thing?

You have no doubt heard of the "law of averages." Laymen use this term rather loosely. There is really no "law" governing averages. Nevertheless, we will see that we can trust an average more than we can trust a single observation. For example, suppose we find the average of the four measurements of the index of refraction of water made in Laboratory Activity 28. If there are no errors built into the experimental design, statisticians have found that this average will be closer to the true index of refraction much more often than any single measurement will be. In fact, the more measurements involved in the calculation of the average, the closer the average tends to be to the true value.

Recall some of our previous discussions of uncertainty. Uncertainty is unavoidable in any measurement. However, it may be reduced by measuring in a more precise manner. For example, one could use an instrument known as a refractometer, which gives very precise measurements of the index of refraction. In this lesson we consider another method of improving measurements--the method of making several independent measurements and averaging the results. This begins our consideration of the "statistical treatment" of data. It is difficult to overemphasize the importance of this technique. Professional scientists and laboratory technicians must have a solid basic knowledge of statistics if they are to perform their jobs well.

Statisticians have their own word for average. They call it the "mean." Since this is a mathematics course, we will also use this term from now on. The mean of a set of numbers is the sum of the numbers divided by the number of numbers in the set. For example, the mean of the set of numbers $\{1, 2, 3, 4\}$ is

$$\frac{1 + 2 + 3 + 4}{4} = 2.5.$$

The mean of a set of numbers is usually signified by a letter with a bar over it. For our example, we write

$$\bar{x} = 2.5$$

and read it as "ecks bar equals 2.5." We may also use summation notation to define the mean. If $\{x_i\}$ is a set of n numbers, then the mean, $\bar{x}$, of that set of numbers is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

This is read, "Ecks bar equals the sum ($\sum$) of ecks sub i (x_i) for i = 1 to n, divided by n." Roughly translated into English, this is, "Add all the x_i's and divide by n." Formulas such as this are convenient because n may be a large number and it would be cumbersome to write out the entire sum.

You will calculate the mean for each of two sets of optics data. These are the four measurements of n_{water} and the four measurements of $n_{glycerine}$ made in the Science Laboratory.

X3-2 The Mean Deviation

We have discussed the idea that the mean is more representative of a group of numbers than any single number selected from the group. In this section we will consider one measure of how far the numbers are from the mean, the "mean deviation." Consider the following two sets of numbers, $\{15, 30, 45\}$ and $\{27, 30, 33\}$. For both sets of numbers, $\bar{x} = 30$. However, the members of the first set are much more widely spread than the members of the second set. The spread is important because we might be more confident with data like the second set than the first set. One measure of spread that statisticians have developed is the mean deviation. The mean deviation for a set of numbers is the sum of the absolute values of the deviations of the numbers from the mean of the set, divided by the number of numbers in the set. For example, for $\{15, 30, 45\}$

$$\begin{aligned}\text{Mean Deviation} &= \frac{|15 - 30| + |30 - 30| + |45 - 30|}{3} \\ &= \frac{15 + 0 + 15}{3} \\ &= 10\end{aligned}$$

For {27, 30, 33}

$$\text{Mean Deviation} = \frac{|27 - 30| + |30 - 30| + |33 - 30|}{3}$$

$$= \frac{3 + 0 + 3}{3}$$

$$= 2$$

Notice that the group that is more widely spread out has the larger mean deviation. The larger the deviation the greater the spread.

In summation notation the mean deviation is

$$\frac{\sum_{i=1}^{n} |x_i - \bar{x}|}{n}$$

where $\{x_i\}$ is a set of n numbers and $\bar{x}$ is the mean of the set. This is read as, "The sum of the absolute deviations from the mean of x_i for i = 1 to n, divided by n." Translated, this becomes, "Add all the absolute values of the deviations from the mean and divide by n."

X3-3 More Error Again

Back in the beginning of Unit II we introduced some techniques and ideas associated with error. First we estimated the uncertainty at each step along the way. Then we used some mathematical techniques to estimate the effect of each of these uncertainties on the final result. The uncertainty estimate that we obtained was of the "worst case" variety. It represented the outer limits of what was possible. We ignored the possibility that one error might tend to cancel out another error. To illustrate the possibilities, consider the following laboratory procedure.

1. Weigh beaker
2. Add chemical
3. Weigh beaker plus chemical

To find the mass of the chemical we subtract the mass of the beaker from the combined mass of the beaker plus chemical. Since absolute uncertainty is additive under the operation of subtraction, the uncertainty of the difference is the sum of the uncertainties of the two measurements (.10 g + .10 g = .20 g). The table below lists some possibilities.

	Measurements		
	Elmo Elbo	Ima Goodwin	True Values
Mass of beaker plus chemical	98.90 ± .10 g	98.85 ± .10 g	98.80 g
Mass of beaker	53.70 ± .10 g	53.82 ± .10 g	53.80 g
Mass of chemical	45.20 ± .20 g	45.03 ± .20 g	45.00 g

Under the "Measurements" column are the attempts of two Biomed students to measure the masses involved. Now look at the numbers in the "True Values" column. Suppose that these numbers were obtained by a very accurate balance. For the sake of argument, we will consider them to be exact. What we are going to do now is examine the pattern of error in Elmo's work and also in Ima's work. We will then compare the two patterns.

Now look at Elmo's measurements. The midpoints of the ranges of uncertainty represent Elmo's best effort to measure the true masses. Let us examine the pattern of errors in his estimates. The 98.90-g measurement differs from the true mass of 98.80 by the maximum amount possible, .10 g. The same is true of the 53.70-g estimate. Notice also that the 98.90 estimate is high and the 53.70 estimate is low. The difference between 98.90 and 53.70 is 45.20 g, a result which differs from the true mass of 45.00 by the maximum possible amount, .20 g. What did Elmo have to do to get a result which differed from the true mass by the maximum amount? First, he had to err by the maximum amounts when he measured the two masses. Second, both measurements had to be off in opposite directions in order to produce an extreme result. Somehow, though, extreme results are to be expected when Elmo does the measuring.

On the other hand, let us consider the measurements obtained by Ima Goodwin. The 98.85 midpoint is higher than the true mass of 98.80 g, but not by the maximum this time. The 53.82 g midpoint is also high, by .02 g. When 53.82 g is subtracted from 98.85 g the difference is 45.03 g. Notice that this result is much closer to the true mass than Elmo's result. What had to happen for Ima to get a more accurate result than Elmo? First, Ima's measurements did not differ from the true mass by the maximum possible amount. Second, Ima got lucky. Her errors were in directions that tended to cancel each other when subtraction occurred. The opposite would have happened if the masses had been added. Errors that tend to cancel each other are called "compensating" errors. They don't always happen. You can't depend on them, but they do occur quite often.

Which pattern of error do you think is more likely to happen? Elmo's pattern, where all of his measurements were off by the maximum amount and also in the proper direction to produce an extreme result? Or Ima's pattern, where her measurements were off, but not by the maximum amounts? Your intuition is good if you chose Ima's pattern as the more likely pattern.

When we use phrases such as "likely to happen" we are getting into the subject of probability. The whole topic of statistics is based upon ideas associated with probability. TV weather forecasters commonly use probability when they say "There is an 80% <u>chance</u> of rain tomorrow." What they mean is, "When conditions are like they are today, it rains the next day in 80% of the cases." In the cases of both rain and patterns of error we cannot predict exactly what will happen. But we can say what outcome is most likely.

You have calculated the index of refraction of water from four pairs of measured angles. Suppose that in one pair of angles you measured both angles a little high. In another pair, both angles were measured a little low. In still another pair, one angle was measured high and another angle was measured low. What actually happens with your particular pattern of errors will be governed partly by chance, randomness, luck, etc. As a consequence of the errors in your angle measurements, your four values of n_i will not agree exactly. They will be spread out. How widely they are spread out is partly a function of the effect of chance measurement errors on your results. The mean deviation measures the amount of spreading of the n_i and therefore provides an estimate of the effect of random measurement errors on your results.

PROBLEM SET X3:

1. The set of numbers {1,2,3} is given.

   a. Find the mean by calculating $\overline{x} = \frac{1 + 2 + 3}{3}$.

   b. Find the mean deviation by calculating $\frac{|1 - \overline{x}| + |2 - \overline{x}| + |3 - \overline{x}|}{3}$

2. Find the mean and the mean deviation for the set {1,2,3,4,5}.

3. Find the mean and the mean deviation for the set {1,2,3,4,5,6,7}.

Two students are measuring the index of refraction of a sugar solution. Each one makes four measurements. The results are shown below.

Measurements

Student A	1.41	1.38	1.37	1.40
Student B	1.43	1.39	1.38	1.40

4. Find the mean and the mean deviation for the "Student A" data.

5. Find the mean and the mean deviation for the "Student B" data.

6. How do the mean deviations for the two students compare?

SECTION X4: STANDARD DEVIATION AND WEIGHTED MEAN

X4-1 The Standard Deviation

In the previous section we considered the problem of estimating the effect of randomness on our experimental results. The mean deviation was presented as one way to estimate the size of the effect of randomness. In this section we will consider yet another measure of spreading, the standard deviation. Statisticians are especially fond of this particular measure of deviation. They have shown that the standard deviation is sometimes very useful in evaluating scientific (and other kinds of) data and also useful for making predictions. In this section we will use the standard deviation in calculating the "weighted mean" for the Index of Refraction data. We will talk about the weighted mean in a little more detail later on. We mention it here only because it is the reason that we are interested in the standard deviation.

The first step involved in finding the standard deviation is the calculation of a quantity called the variance, s^2. If $\{x_i\}$ is a set of n numbers,

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

The first step in the calculation of the variance is to sum all of the squares of the deviations from the mean. Compare this to the mean deviation. The first step was to sum all of the absolute deviations from the mean. To find the variance, first we add up all of the squares of the deviations and then divide by (n - 1).

Consider the following sample calculation of the variance, s^2.

CALCULATION OF THE VARIANCE, s^2

$$s^2 = \frac{1}{n-1} \left[\sum_{i=1}^{n} (x_i - \bar{x})^2 \right]$$

$\bar{x} = 1.34$

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
$x_1 = 1.40$	.06	.0036
$x_2 = 1.31$	-.03	.0009
$x_3 = 1.35$	.01	.0001
$x_4 = 1.30$	-.04	.0016
	sum:	$.0062 = \sum_{i=1}^{4} (x_i - 1.34)^2$

ERIC

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n}(x_i - \overline{x})^2\right], \; n = 4$$

$$s^2 = \frac{1}{4-1}\left[\sum_{i=1}^{4}(x_i - 1.34)^2\right]$$

$$s^2 = \frac{1}{3}(.0062)$$

$$s^2 \simeq .0021$$

This result should be restated in scientific notation. The exponent of 10 should be even. The reason for this will be obvious a little later.

$$s^2 \simeq 21 \times 10^{-4}$$

Now, the standard deviation is the square root of the variance; therefore,

$$s \simeq \sqrt{21 \times 10^{-4}}$$

Since the exponent of 10 is even, we find its square root easily.

$$\sqrt{10^{-4}} = 10^{-2}$$

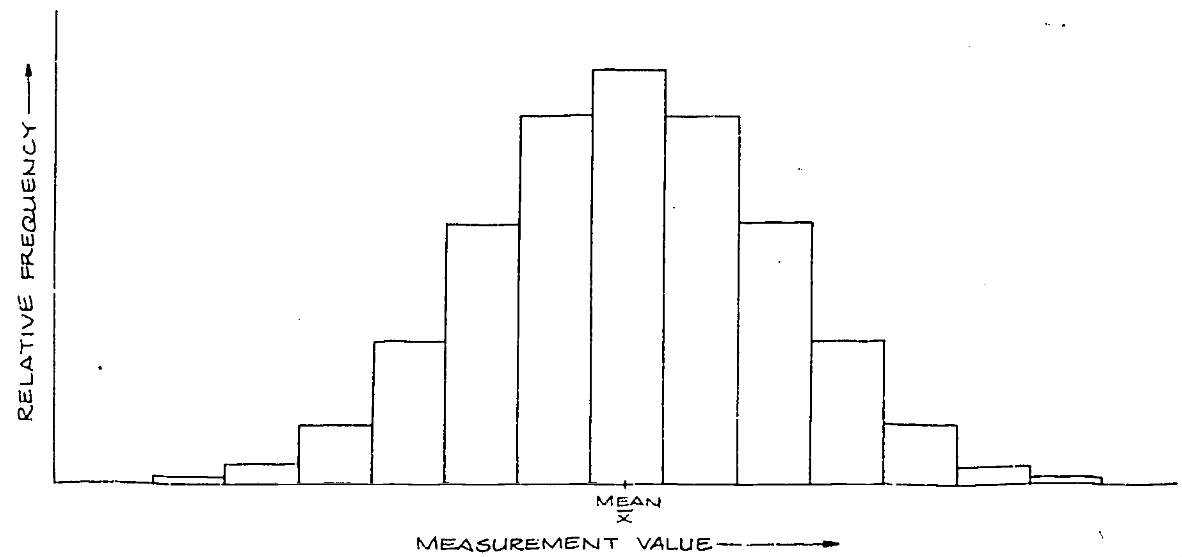
By consulting the square root table at the end of this book, we find that $\sqrt{21} \simeq 4.6$. Consequently,

$$s \simeq 4.6 \times 10^{-2}$$

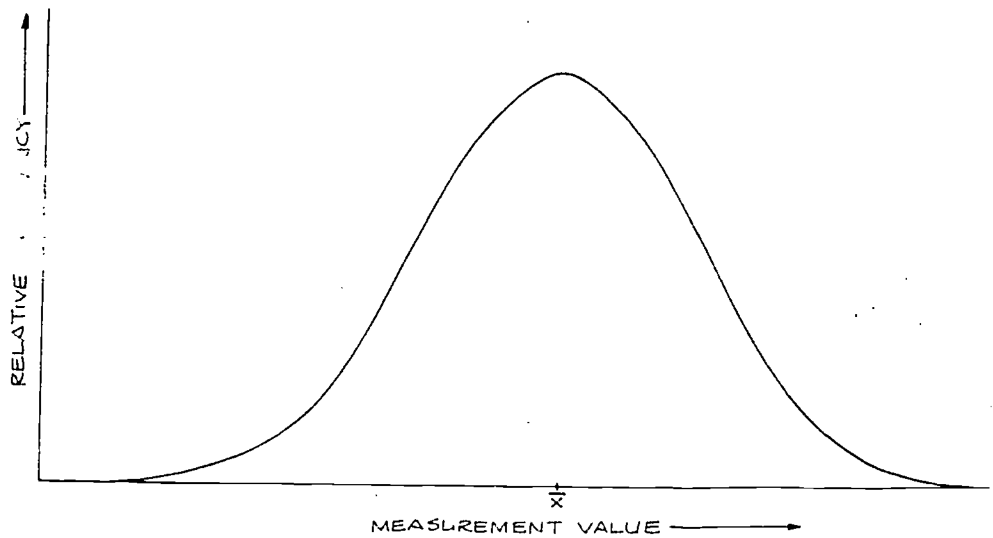
In this lesson you will calculate the standard deviation for your four measurements of the refractive index of water.

X4-2 Standard Deviation and Frequency Distributions

Suppose that instead of four measurements of the refractive index of water, you had made a hundred or a thousand. What sort of pattern might you expect to see in this larger data set? If you made all the measurements carefully it seems reasonable that most of them would cluster around a mean value. However, since chance errors always occur, most of the values would deviate from the mean. It seems reasonable that large deviations would happen rarely and smaller ones more often. One way to visualize the results would be to construct a relative-frequency histogram, just as you did in the chi square lessons for your coin-tossing data. The result would look something like the histogram below. As you can see, the measurements with the highest relative frequency are those close to the mean. Measurements that are far from the mean are much less frequent.

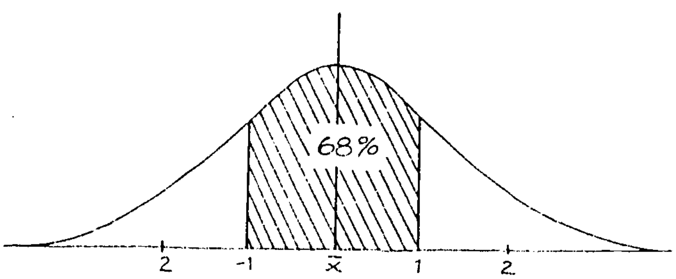


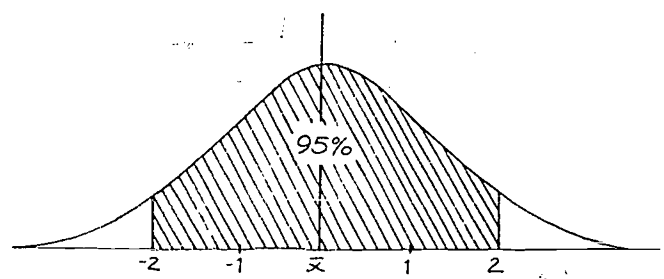
Suppose that we now draw a smooth curve with the same shape as the histogram. The result is shown below.



If you open a statistics book you will see many curves with this appearance. They are called normal distributions. Most sets of data produce distributions with this general form. They are not always so symmetrical but they have the same general appearance.

We might now ask, how does the standard deviation relate to the normal distribution? The answer is that it can tell us something about how the measurements will be distributed without our having to draw the distribution curve at all. It turns out that about 68% of all measurements are less than one standard deviation from the mean. About 95% of the measurements are less than two standard deviations from the mean. We can show these relationships graphically on a normal distribution with the horizontal axis scaled in units of standard deviations.





This relationship is helpful to a statistician in judging how unusual a given measurement is. For example, suppose the mean is 20, the standard deviation is 5 and we have a measurement of 9. We reason that 9 is 11 units from the mean. Two standard deviations is 10 units from the mean. Therefore the measurement is more than two standard deviations from the mean. Only 5% of measurements are more than two standard deviations from the mean. Finally, only half of these will be two standard deviations below the mean. Therefore we know that a measurement this low should happen less than 2.5% of the time.

XI-3 Weighted Means

Suppose that the Biomed class took a test. Further, suppose that the females averaged 93 and the males averaged 87. We will ignore the likelihood that this would or would not happen. It is unimportant for our present purposes. What is the mean score for the entire class? Should we just add 93 and 87 and divide by two? No, because it ignores the possibility that there may be different numbers of males and females. We need to know the numbers of males and females to find the mean for the entire class. Suppose then that there are 10 males and 20 females. We may find the class mean in the following way. An explanation of the meaning behind our method follows the example.

$$\bar{x} = \frac{(10 \cdot 87) + (20 \cdot 93)}{10 + 20}$$

$$= \frac{870 + 1{,}860}{30}$$

$$= \frac{2{,}730}{30}$$

$$= 91$$

Notice what we did. In the numerator we multiplied the male average by the number of males $(10 \cdot 87)$, and multiplied the female average by the number of females $(20 \cdot 93)$ and added the two products. We then divided this sum by the total number of students $(10 + 20)$. The result is the same as we would have obtained by adding all of the individual scores and dividing by the total number of students. You may be asking yourself about now, "Why is this new method better than the way we would normally solve such a problem?" Actually the new method is often quicker.

In the previous example, we can call the "weight" of the female scores 20 and the weight of the male scores 10. Now we can say that we multiplied each mean by its weight, added these two products, and then divided this sum by the total weight. In symbolic form, if $\{\bar{x}_i\}$ is a set of n means and w_i is the weight of $\bar{x}_i$, then

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i \bar{x}_i}{\sum_{i=1}^{n} w_i}$$ or, for our specific example,

$$\bar{x}_w = \frac{(10 \cdot 87) + (20 \cdot 93)}{(10 + 20)}$$

This process for finding the weighted mean for a set of means is general. We may apply it in many situations that differ from the simple example that we have presented. It often happens that we will want to find the average of several measurements but we won't want to give all of the numbers the same weight.

Now consider the problem of how to combine the index of refraction results from two lab stations. There is a mean refractive index and a standard deviation for each station. Should we just add both means and divide by two? No, because it neglects the possibility that the standard deviations are different. The mean with the smaller standard deviation is a "better" estimate. It is better because the effect of randomness is less. If we added the two means and divided by two, we would be saying that one estimate is just as good as another. This isn't true. Somehow we must give more weight to the better estimate. Suppose we assign a weight to each mean, as we did above. We let the weight equal the reciprocal of the standard deviation. As the standard deviation decreases the weight increases. This does what we want it to. It gives more weight to the better estimate. The only remaining problem is how to combine the means and their weights in order to get a meaningful result. Consider this example.

$\bar{x}_1 = 15 \qquad s_1 = .5$

$\bar{x}_2 = 18 \qquad s_2 = 1$

The weight for $\bar{x}_1$ is 2 because $\frac{1}{.5} = 2$. The weight for $\bar{x}_2$ is 1 because $\frac{1}{1} = 1$. Therefore,

$$\bar{x}_1 = 15 \qquad w_1 = 2$$

$$\bar{x}_2 = 18 \qquad w_2 = 1$$

This says that $\bar{x}_1$ should receive twice the weight that $\bar{x}_2$ receives. One way to arrange this is to add twice 15 and 18 and divide by three.

$$\begin{array}{r} 15 \\ 15 \\ +\ 18 \\ \hline 48 \end{array} \qquad \begin{array}{r} \bar{x}_1 \\ \bar{x}_1 \\ +\ \bar{x}_2 \\ \hline 2\bar{x}_1 + \bar{x}_2 \end{array}$$

$$\frac{48}{3} = \frac{2\bar{x}_1 + 1\bar{x}_2}{2 + 1} = 16$$

We call 16 the "weighted mean" of 15 and 18.

Notice what we did. We multiplied each mean by its weight, added the products and divided by the total weight just as we did before. However, the weight of each mean is found by a different method. The weight is the reciprocal of the standard deviation. For your reference, we repeat the general formula for the weighted mean below. This formula includes the procedure for finding the weight.

If $\{\bar{x}_i\}$ is a set of n means, then

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i \bar{x}_i}{\sum_{i=1}^{n} w_i}$$

where $w_i = \frac{1}{s_i}$

You will calculate the weighted mean for different sets of index of refraction data. Below is a problem which more nearly represents the situation you will face.

DATA:

$$\bar{x}_1 = 1.35 \qquad s_1 = .02 \qquad w_1 = \frac{1}{s_1} = 50$$

$$\bar{x}_2 = 1.30 \qquad s_2 = .04 \qquad w_2 = \frac{1}{s_2} = 25$$

$$\bar{x}_w = \frac{w_1\bar{x}_1 + w_2\bar{x}_2}{w_1 + w_2}$$

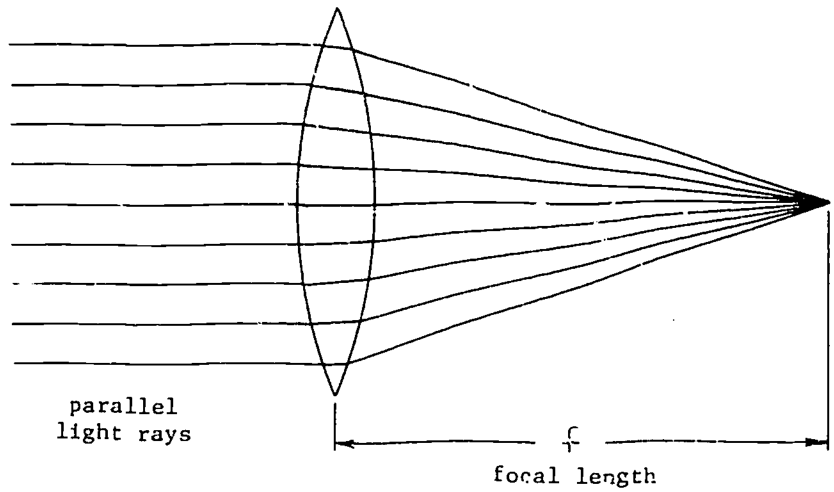
$$= \frac{(50 \cdot 1.35) + (25 \cdot 1.30)}{50 + 25}$$

$$= \frac{67.5 + 32.5}{75}$$

$$\approx 1.33$$

SECTION Y1: CALCULATION OF FOCAL LENGTH

In Laboratory Activity 30 you measured the image distances and object distances for lenses with different refractive indexes and different radii of curvature. The refractive index is changed by filling the watch glass lenses with different concentrations of sugar (sucrose) solutions. In this lesson you will calculate the focal lengths corresponding to each concentration of sugar solution. Recall that the focal length of a lens is the distance at which parallel light rays will converge.



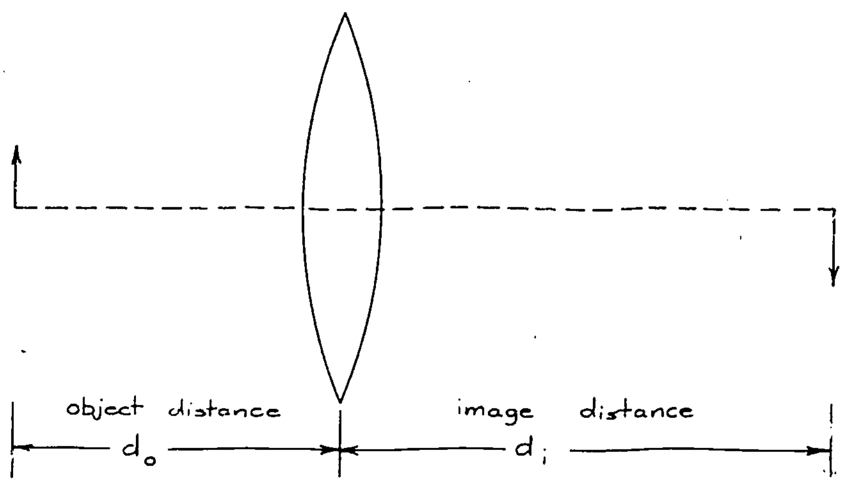
The focal length of a lens is related to the image distance and object distance by the equation

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

where f = focal length,

d_o = object distance,

d_i = image distance.



This form of the equation is not very convenient to use. It may be transformed into a more convenient form as follows.

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

Multiply $\frac{1}{d_i}$ by $\frac{d_o}{d_o}$ and $\frac{1}{d_o}$ by $\frac{d_i}{d_i}$. Since $\frac{d_o}{d_o} = 1$ and $\frac{d_i}{d_i} = 1$ the equality is preserved.

$$\frac{1}{f} = \left(\frac{1}{d_o} \cdot \frac{d_i}{d_i}\right) + \left(\frac{1}{d_i} \cdot \frac{d_o}{d_o}\right)$$

$$\frac{1}{f} = \frac{d_i}{d_o d_i} + \frac{d_o}{d_i d_o}$$

Since both fractions on the right have the same denominator, the numerators may be added.

$$\frac{1}{f} = \frac{d_i + d_o}{d_i d_o}$$

We may now take the reciprocal of each side without loss of the equality.

$$f = \frac{d_i d_o}{d_i + d_o}$$

This equation states that the focal length of a lens is the product of d_i and d_o divided by the sum of d_i and d_o. This is the form of the equation you should use.

Below is a sample of the experimental results obtained by Ms. Ima Sweetlens. Ima has done everything correctly so far. She has subtracted the object distance from the screen position in order to get the image distance.

	0% Sucrose		
Screen position (cm)	50.0	65.6	84.9
d_o: object distance (cm)	30.0	50.0	70.0
d_i: image distance (cm)	20.0	15.6	14.9
Experimental focal length (cm)	f_1	f_2	f_3

Ima likes to start with the hardest things first. She sets out to calculate f_3.

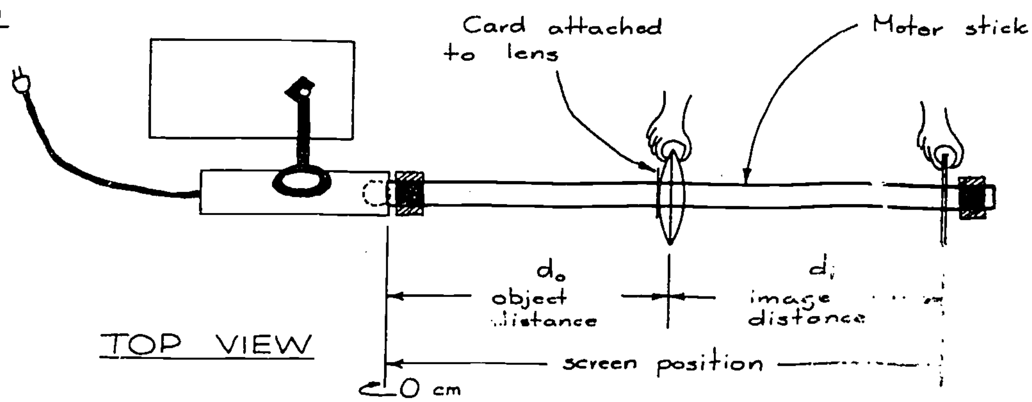
d_i = 14.9 cm

d_o = 70.0 cm

$$f_3 = \frac{d_i d_o}{d_i + d_o}$$

$$= \frac{(14.9)(70.0)}{(14.9 + 70.0)}$$

$$= \frac{(14.9)(70)}{84.9}$$



If Ima had a calculator or slide rule she would use one of these helpful tools. She doesn't. If you do, you should use it. State your result with an uncertainty of .05 cm. Since Ima doesn't have a calculator, she rounds everything in sight to the nearest whole number.

$$f_3 = \frac{(14.9)(70)}{84.9}$$

$$f_3 \approx \frac{(15)(70)}{85}$$

It is important not to do any rounding until you have calculated the sum in the denominator. Notice that the denominator is numerically equal to the screen distance. Some 5's may be canceled in the quotient for f_3.

$$f_3 \approx \frac{\overset{3}{\cancel{(15)}}(70)}{\underset{17}{\cancel{85}}}$$

$$f_3 \approx \frac{210}{17}$$

```
      12.3
17/210.00
   17
    40
    34
     60
     51
      9
```

Ima reports f_3 as 12.4 cm because 9 is more than half of 17.

When Ima has completed the calculations for f_1, f_2 and f_3, she calculates $\bar{f}$ for 0% sugar (water) solution, which should be rounded to the nearest .1 cm (uncertainty of .05 cm).

You should follow Ima's example and calculate nine f_i's and three $\bar{f}$'s.

PROBLEM SET Y1:

1. a. The experimental focal-length equation for thin lenses is $\frac{1}{f} = ?$

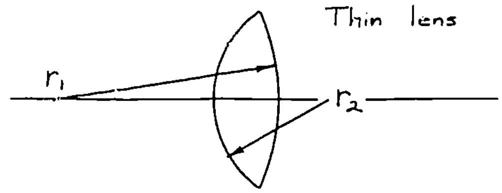
(1) $\frac{1}{d_o} + \frac{1}{d_i}$ (2) $d_o + d_i$ (3) $\frac{1}{d_i} - \frac{1}{d_o}$

b. When the focal-length equation is solved for d_i, the image distance, the result is: $d_i = ?$

(1) $\frac{d_o + d_i}{d_o d_i}$ (2) $\frac{d_o \cdot f}{d_o - f}$ (3) $\frac{1}{d_o} - f \cdot d_o$

c. In a focal-length experiment a thin lens (f = 12 cm) was placed above the 30-cm mark on the meter stick. (The light source was above the zero mark.) What was the image distance d_i? (Refer to Part b.)

2. The theoretical focal-length formula of a thin lens is $\frac{1}{f} = (n - 1)(\frac{1}{r_1} + \frac{1}{r_2})$.



a. n is the index of (?) of the lens.

b. r_1 and r_2 are the radii of (?) of the lens.

c. In a symmetrical lens $r_1 = r_2$. This reduces the theoretical focal-length equation to:

(1) $\frac{1}{f} = (n - 1)r_1$ (2) $\frac{1}{f} = (\frac{1}{r_1})(n - 1)$ (3) $\frac{1}{f} = (n - 1)(\frac{2}{r_1})$

d. Solve Part c for f.

3. Find the radius of curvature for a symmetrical thin lens with a focal length of 15 cm and an index of refraction of 1.3.

4. Both faces of a thin lens in a camera have a radius of curvature of 4.0 cm and a focal length of 3.8 cm. What is the index of refraction of the lens to the nearest .01?

5. Before the heavyweight championship of the world, the champ glowered at the challenger who was standing 80 cm away. (In the eye the retina or "image screen" is located 2 cm behind the eye-lens.)

a. What is the object distance d_o?

b. What is the image distance d_i?

c. Find the focal length of the champ's eye-lenses when he glared at his opponent. (Assume that the eye behaves like a symmetrical lens.)

6. a. So that the eye can focus on near and far objects, the focal length of the eye-lens changes. How far away is the object focused on when f = 1.8 cm? (Remember, d_i = 2 cm.)

b. When the eye focuses on near objects the focal length:

(1) increases (2) decreases (3) remains constant

7. A camera with a thin lens has a focal length of 4.5 cm. It is used to photograph an angry bumblebee 50.0 cm away. The lens position should be adjusted so that the lens-to-film distance is (?) cm. (The film is the image screen.)

SECTION Y2: STANDARD DEVIATION AND FOCAL LENGTH

Y2-1 Calculation of the Standard Deviation of Each Set of Three Focal-Length Measurements

Ima completed her Data Sheet to the point indicated by the arrow. When her instructor returned Data Sheet 30 she quickly calculated the deviations from the mean and squared them.

	0% Sucrose		
Screen position (cm)	50.0	65.6	84.9
d_o: object distance (cm)	30.0	50.0	70.0
d_i: image distance (cm)	20.0	15.6	14.9
Experimental focal length (cm)	f_1	f_2	f_3
	12.0	11.9	12.4
Completed as of Lesson Y1 → Mean focal length, $\overline{f}$ (cm)	12.1		
Deviations from mean focal length: $= f_i - \overline{f}$ (cm)	-.1	-.2	.3
$(f_i - \overline{f})^2$ (cm^2)	.01	.04	.09
Standard deviation: s (cm)			

She summed the squares of the deviations to get .14. She substituted this sum into the equation

$$s = \sqrt{\frac{\sum (f_i - \overline{f})^2}{n - 1}}$$

$$s = \sqrt{\frac{.14}{2}}$$

$$s = \sqrt{.07}$$

$$s = \sqrt{7 \times 10^{-2}}$$

$$s = \sqrt{7} \times 10^{-1}$$

She consulted the Square Root Table to find that $\sqrt{7} \approx 2.6$; therefore

$$s \approx 2.6 \times 10^{-1}$$

$$\text{or } s \approx .26$$

You should perform similar calculations for each of your three sets of data.

Y2-2 A Way to Predict the Focal Length of a Lens

It is possible to predict the focal length of a symmetrical lens from its radius of curvature and its refractive index. The focal length of a symmetrical lens is related to these two variables by the equation at the top of the next page.

$$f_{\theta} = \frac{R}{2(n - 1)}$$

where n = index of refraction of the lens,

R = radius of curvature of the lens,

f_{θ} = theoretical focal length of the lens.

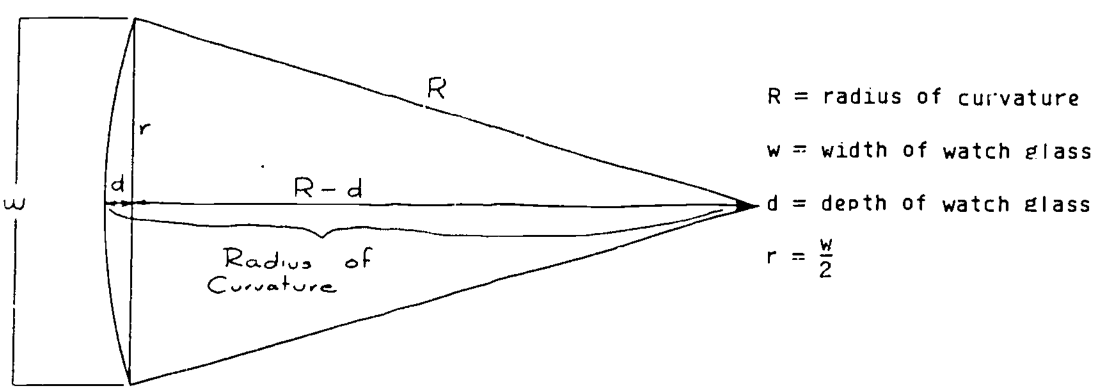
For people who think that physics is a beautiful subject, this particular formula is especially pleasing. Let's see what it says. First of all, it says that the greater the radius of curvature of the lens (greater R), the longer the theoretical focal length. For example, a glass BB has a small radius of curvature. It also has a short focal length. On the other hand, a flat plane of glass will have an infinite radius of curvature. The focal length will also be infinite. This means that a flat plane of glass will not focus light, and we know this is also true.

Consider now the (n - 1) term. The "1" represents the refractive index of air. The term (n - 1) is then the difference between the refractive index of the lens and the refractive index of air. The greater this difference, the more light will be bent upon entering the lens, which will result in a shorter focal length. In the formula the (n - 1) term is in the denominator. An increase in the value of the denominator will decrease the value of the fraction. This is the same as saying that the theoretical focal length will decrease with an increase in the denominator. Again this agrees with our previous knowledge and intuition.

Now it's time to look at the "2." It stands for the fact that the lenses you experimented with had two sides. Since the 2 is in the denominator it says that the focal length will be half of what it would be if there were only one curved side to the lens. In summary, the surprising thing about all of this is that no unexplainable constant is sitting in front of the $\frac{R}{2(n - 1)}$. Every term in the expression is explainable on a completely intuitive level.

You will calculate the radius of curvature of your watch glasses from your measurements of their width and depth. Chemists have determined the refractive indexes of the different sugar solutions that you used. These are the two pieces of information that you need to calculate a predicted value for the focal length of your watch-glass lens. If you are lucky, your predictions will agree rather closely with the $\bar{f}$'s for each of the different solutions. However, do not get your expectations too high. The watch-glass lenses are very crude items. It is too much to expect them to be well behaved.

The following is a diagram which shows the important quantities.



You measured w and d in Laboratory Activity 30. Notice that $r = \frac{w}{2}$ or half the width of the watch glass. The Pythagorean Theorem relates r and d to R as follows.

$$R^2 = (R - d)^2 + r^2$$

When this equation is solved for R, we get

$$R = \frac{r^2 + d^2}{2d}$$

The following are some sample measurements of a watch glass.

Watch glass depth: $d = 1.7$ cm

Watch glass width: $w = 10.0$ cm

$$r = \frac{w}{2} = 5.0 \text{ cm}$$

To find R we substitute the numerical values of d and r into the preceding equation.

$$R = \frac{(5.0)^2 + (1.7)^2}{2(1.7)}$$

$$= \frac{25 + 2.89}{3.4}$$

$$\approx \frac{27.9}{3.4}$$

$$R \approx 8.2 \text{ cm}$$

To find f_θ we substitute our value of R into the equation.

$$f_\theta \approx \frac{R}{2(n - 1)}$$

If water was in the lens, then $n \approx 1.333$ and

$$f_\theta \approx \frac{8.2}{2(1.333 - 1)}$$

$$\approx \frac{8.2}{2(.333)}$$

Notice that $.333 \approx \frac{1}{3}$; thus

$$f_\theta \approx \frac{8.2}{\frac{2}{3}}$$

$$\approx 8.2 \cdot \frac{3}{2}$$

$$\approx 12.3 \text{ cm}$$

This result gets entered at the bottom of Data Sheet 30 as follows:

Theoretical focal length, f_θ (cm)	12.3

Computations of the above type must be understood thoroughly by anyone who is making lenses. An optician, who grinds lenses for glasses, must have a good working knowledge of these formulas.

REVIEW PROBLEM SET Y3:

1. Recall the formula $n = \frac{\sin \alpha}{\sin \beta}$.

   a. n is called the (?)

   b. α is called the (?)

   c. [illegible] called the (?)

2. Suppose that a hydrochloric acid solution has an index of refraction of 1.166. What is the speed of light in the solution? (The speed of light in air is 3.00×10^8 m/sec.)

In Problems 3 through 7 refer to the table at the right, which gives indexes of refraction for four sulfuric acid solutions.

Solution	Density	Index of Refraction
A	1.028	1.339
B	1.221	1.370
C	1.632	1.425
D	1.811	1.437

3. As the solution density increases, the index of refraction (increases, decreases).

4. Light will travel slowest in solution (?)

5. Light will travel fastest in solution (?)

6. Suppose that an experiment with one of the solutions yields the results, angle of incidence = 34° and angle of refraction = 24°. Which solution was used in the experiment?

7. Suppose α = 60° and β = 37°. Which solution was used?

In Problems 8 through 14, refer to the table at the right, which shows the data obtained in an experiment carried out by two students.

Student	x_1	x_2	x_3
A	8	10	12
B	5	10	15

8. Which student's results are most widely scattered (spread out)?

9. Find $\overline{x}$ for the Student A data.

10. Find the mean deviation for the Student A data. Use the formula

$$\text{mean deviation} = \frac{\sum_{i=1}^{n} |x_i - \overline{x}|}{n}$$

11. Find $\overline{x}$ for the Student B data.

12. Find the mean deviation for the Student B data.

13. The mean deviation was larger for the data of Student (?)

The table at the right shows the data from two trials of an experiment.

Trial	x_1	x_2	x_3
1	5	7	12
2	5	10	18

14. The data is more widely spread in Trial (?)

15. Calculate the mean for Trial 1.

16. Find the standard deviation for the Trial 1 data (to the nearest hundredth). Use the formula

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n - 1}}$$

17. Find the mean for the Trial 2 data.

18. Find the standard deviation for the Trial 2 data (nearest hundredth).

19. The standard deviation was larger for Trial (?)

20. The table at the right shows the means and standard deviations of three experiments on measuring index of refraction.

Experiment	$\overline{x}$	s
1	1.60	.05
2	1.57	.02
3	1.52	.01

Compute the weighted mean $\bar{x}_w$ according to the formula

$$\bar{x}_w = \frac{\sum_{i=1}^{n} w_i \bar{x}_i}{\sum_{i=1}^{n} w_i}, \quad w_i = \frac{1}{s_i}$$

21. The experimental focal length equation for thin lenses is

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

a. d_o is called the (?) b. d_i is called the (?)

c. What is the result when the above equation is solved for f?

d. What is the result when the above equation is solved for d_i?

22. Suppose the focal length, f, of a thin lens is 10 cm. What will the image distance be when the object distance is

a. 20 cm? b. 110 cm? c. 1,010 cm?

d. As d_o gets larger, f and d_i are getting (closer together, farther apart).

e. If the light rays from the object are parallel when they hit the lens then d_i = (?)

23. The Lens Maker's Formula for a thin lens is $\frac{1}{f} = (n - 1)(\frac{1}{r_1} + \frac{1}{r_2})$. Suppose a lens has an index of refraction of 1.5 and that the faces have radii of curvature of 20 cm and 30 cm. Find the focal length of the lens.

24. In the case of a symmetrical thin lens $r_1 = r_2$ and the Lens Maker's Formula takes the form

$$\frac{1}{f} = (n - 1)(\frac{2}{r})$$

a. Suppose that a symmetrical lens has a radius of curvature of 16 cm and a focal length of 20 cm. What is the index of refraction of the lens?

b. At what speed does light travel through the lens? (The speed of light in air is 3.00×10^8 m/sec.)

25. Suppose that a symmetrical thin lens has a radius of curvature of 12 cm. Suppose an object is 30 cm from the lens and the image distance is 15 cm. Compute the index of refraction of the lens.

SECTION Z: EVALUATING THE VISION-SCREENING DATA

Z-1 Snellen Accuracy

You have administered the Snellen Visual Acuity Test to some primary school children. Some children will "pass" the test; others will "fail." You will recommend that the children who failed the test be referred for professional attention if they are not receiving it already. There are a couple of important questions connected with this that should be answered. They are, "How good is the Snellen Test? How many children passed the test who should have failed? How many were referred who didn't need any professional care at all?" Not surprisingly, the answers are related to the cutoff chosen for the Snellen Test. Suppose one group of Biomed students decided to refer all children who had 20/40 vision or worse. If they did their screening well, then their results would be similar to the ones shown in the table below.

Sample of 100 Second-Graders

Cutoff	Number of Passes	Number of Referrals	Number of Those Referred Who Do Not Need Help	Number of Children Who Passed with Undetected Vision Problems
20/40	92	8	0	9

Notice that in this particular case, 92 children "passed." Now look at the far-right column. It says there that 9 of the 92 who passed the test had undetected vision problems. Of course, it would require more sophisticated visual testing to find this out.

We are going to define some new words. The 92 children who passed are called "negatives" by social scientists and statisticians. They are called negatives because the question, "Does this child have a visual acuity problem?" was answered in the negative. The nine children who had undetected vision problems are called "false negatives." This is because for them a negative answer is wrong or false.

Now look at the third column of the table above. Notice that 8 children were referred. These children are called "positives." The fourth column shows that of the students who were referred, there were none who did not need care. In other words, all eight referrals actually needed care. We say that there were no "false positives."

So far, so good? Well, not quite. We should be concerned about those 9 false negatives--the ones with undetected visual problems. Nobody who sincerely wants to find all of the children who need care can be happy about those 9. In fact, only 8 were referred. This means that 9 out of 17 or a majority of children with vision problems went undetected. One way to catch more vision problems might be to raise the standards for passing. Let's see what happens when the cutoff point is changes to 20/30.

Cutoff	Number of Passes	Number of Referrals	Number of Those Referred Who Do Not Need Help	Number of Children Who Passed With Undetected Vision Problems
20/30	85	15	3	5

Notice that the number of false negatives is reduced from 9 to 5. This is a desirable result. However, notice that there is a counter-effect. Some children were referred who didn't need care. Fifteen children were referred, and only 12 of them needed care. Three did not need care. These three children are called "false positives." In summary, switching to a 20/30 cutoff has two opposing effects. One effect is to decrease the number of false negatives (desirable). The other is to increase the number of false positives (undesirable).

Which cutoff should be used? We will leave the answer to that question to you and your Science instructor. Opinions differ in the world at large. You and your teacher should make the decision that best suits you.

Z-2 What Does It All Mean?

Your Biomedical class administered the Snellen Test to itself and to a group of primary-school children. The results of these activities will be compared with the results of the Orinda Study. These are a lot of data. You will find the mean, median and mode of each set of data. These statistics will be useful in a consideration of the following questions.

1. Do males have better (or worse) vision than females?

2. Does age have anything to do with visual acuity?

3. Does socioeconomic class have anything to do with visual acuity?

Unfortunately, you will not be able to answer any of these questions with certainty. Statistics is the land of the definite maybe. Any answer that you may agree on for your particular sets of data should be considered to be very, very tentative. More advanced statistical techniques are necessary to reach more conclusive answers.

SQUARE ROOT TABLE

n	√n	n	√n	n	√n	√10n	n	√n	√10n	n	√n	√10n	n	√n	√10n
		50	7.071	100	10.00	31.62	150	12.25	38.73	200	14.14	44.72	250	15.81	50.00
1	1.000	51	7.141	101	10.05	31.78	151	12.29	38.86	201	14.18	44.83	251	15.84	50.10
2	1.414	52	7.211	102	10.10	31.94	152	12.33	38.99	202	14.21	44.94	252	15.87	50.20
3	1.732	53	7.280	103	10.15	32.09	153	12.37	39.12	203	14.15	45.06	253	15.91	50.30
4	2.000	54	7.348	104	10.20	32.25	154	12.41	39.24	204	14.28	45.17	254	15.94	50.40
5	2.236	55	7.416	105	10.25	32.40	155	12.45	39.37	205	14.32	45.28	255	15.97	50.50
6	2.449	56	7.483	106	10.30	32.56	156	12.49	39.50	206	14.35	45.39	256	16.00	50.60
7	2.646	57	7.550	107	10.34	32.71	157	12.53	39.62	207	14.39	45.50	257	16.03	50.70
8	2.828	58	7.616	108	10.39	32.86	158	12.57	39.75	208	14.42	45.61	258	16.06	50.79
9	3.000	59	7.681	109	10.44	33.01	159	12.61	39.87	209	14.46	45.72	259	16.09	50.89
10	3.162	60	7.746	110	10.49	33.17	160	12.65	40.00	210	14.49	45.83	260	16.12	50.99
11	3.317	61	7.810	111	10.54	33.32	161	12.69	40.12	211	14.52	45.93	261	16.16	51.09
12	3.464	62	7.874	112	10.58	33.47	162	12.73	40.25	212	14.56	46.04	262	16.19	51.19
13	3.606	63	7.937	113	10.63	33.62	163	12.77	40.37	213	14.59	46.15	263	16.22	51.28
14	3.742	64	8.000	114	10.68	33.76	164	12.81	40.50	214	14.63	46.26	264	16.25	51.38
15	3.873	65	8.062	115	10.72	33.91	165	12.85	40.62	215	14.66	56.37	265	16.28	51.48
16	4.000	66	8.124	116	10.77	34.06	166	12.88	40.74	216	14.70	46.48	266	16.31	51.58
17	4.123	67	8.185	117	10.82	34.21	167	12.92	40.87	217	14.73	46.58	267	16.34	51.67
18	4.243	68	8.246	118	10.86	34.35	168	12.96	40.99	218	14.76	46.69	268	16.37	51.77
19	4.359	69	8.307	119	10.91	34.50	169	13.00	41.11	219	14.80	46.80	269	16.49	51.87
20	4.472	70	8.367	120	10.95	34.64	170	13.04	41.23	220	14.83	46.90	270	16.43	51.96
21	4.583	71	8.426	121	11.00	34.79	171	13.08	41.35	221	14.87	47.01	271	16.46	52.06
22	4.690	72	8.485	122	11.04	34.93	172	13.11	41.47	222	14.90	47.12	272	16.49	52.14
23	4.796	73	8.544	123	11.09	35.07	173	13.15	41.59	223	14.93	47.22	273	16.52	52.25
24	4.899	74	8.602	124	11.13	35.21	174	13.19	41.71	224	14.97	47.33	274	16.55	52.35
25	5.000	75	8.660	125	11.18	35.36	175	13.23	41.83	225	15.00	47.43	275	16.58	52.44
26	5.099	76	8.718	126	11.22	35.50	176	13.27	41.95	226	15.03	47.54	276	16.61	52.54
27	5.196	77	8.775	127	11.27	35.64	177	13.30	42.07	227	15.07	47.64	277	16.64	52.63
28	5.292	78	8.832	128	11.31	35.78	178	13.34	42.19	228	15.10	47.75	278	16.67	52.73
29	5.385	79	8.888	129	11.36	35.92	179	13.38	42.31	229	15.13	47.85	279	16.70	52.82
30	5.477	80	8.944	130	11.40	36.06	180	13.42	42.43	230	15.17	47.96	280	16.73	52.92
31	5.568	81	9.000	131	11.45	36.19	181	13.45	42.54	231	15.29	48.96	281	16.76	53.01
32	5.657	82	9.055	132	11.49	36.33	182	13.49	42.66	232	15.23	48.17	282	16.79	53.10
33	5.745	83	9.110	133	11.53	36.47	183	13.53	42.78	233	15.26	58.27	283	16.82	53.20
34	5.831	84	9.165	134	11.58	36.61	184	13.56	42.90	234	15.30	58.37	284	16.84	53.29
35	5.916	85	9.220	135	11.62	36.74	185	13.60	43.01	235	15.33	48.48	285	16.88	53.39
36	6.000	86	9.274	136	11.66	36.88	186	13.64	43.13	236	15.36	48.58	286	16.91	53.48
37	6.083	87	9.327	137	11.70	37.01	187	13.67	43.24	237	15.39	58.68	287	16.94	53.57
38	6.164	88	9.381	138	11.75	37.15	188	13.71	43.36	238	15.43	48.79	288	16.97	53.67
39	6.245	89	9.434	139	11.79	37.28	189	13.75	43.47	239	15.46	48.89	289	17.00	53.76
40	6.325	90	9.487	140	11.83	37.42	190	13.78	43.59	240	15.49	48.99	290	17.03	53.85
41	6.403	91	9.539	141	11.87	37.55	191	13.82	43.70	241	15.52	49.09	291	17.06	53.94
42	6.481	92	9.592	142	11.92	37.68	192	13.86	43.82	242	15.56	59.19	292	17.09	54.04
43	6.557	93	9.644	143	11.96	37.82	193	13.89	43.93	243	15.59	49.30	293	17.12	54.13
44	6.633	94	9.695	144	12.00	37.95	194	13.93	44.05	244	15.62	49.40	294	17.15	54.22
45	6.708	95	9.747	145	12.04	38.08	195	13.96	44.16	245	15.65	49.50	295	17.18	54.31
46	6.782	96	9.798	146	12.08	38.21	196	14.00	44.27	246	15.68	49.60	296	17.20	54.41
47	6.856	97	9.849	147	12.12	38.34	197	14.04	44.38	247	15.72	49.70	297	17.23	54.50
48	6.928	98	9.899	148	12.16	38.47	198	14.07	44.50	248	15.75	49.80	298	17.26	54.59
49	7.000	99	9.950	149	12.21	38.60	199	14.11	44.61	249	15.78	59.90	299	17.29	54.68

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n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
300	17.32	54.77	350	18.71	59.16	400	20.00	63.25	450	21.21	67.08	500	22.36	70.71
301	17.35	54.86	351	18.73	59.25	401	20.02	63.32	451	21.24	67.16	501	22.38	70.78
302	17.38	54.95	352	18.76	59.33	402	20.05	63.40	452	21.26	67.23	502	22.41	70.85
303	17.40	55.05	353	18.79	59.41	403	20.07	63.48	453	21.28	67.31	503	22.43	79.92
304	17.44	55.14	354	18.81	59.50	404	20.10	63.56	454	21.31	67.38	504	22.45	70.99
305	17.46	55.23	355	18.84	59.58	405	20.12	63.64	455	21.33	67.45	505	22.47	71.06
306	17.49	55.32	356	18.87	59.67	406	20.15	63.72	456	21.35	67.53	506	22.49	71.13
307	17.52	55.41	357	18.89	59.75	407	20.17	63.80	567	21.38	67.60	507	22.52	71.20
308	17.55	55.50	358	18.92	59.83	408	20.20	63.87	458	21.40	67.68	508	22.54	71.27
309	17.58	55.59	359	18.95	59.92	409	20.22	63.95	459	21.42	67.75	509	22.56	71.34
310	17.61	55.68	360	18.97	60.00	410	20.25	64.03	460	21.45	67.82	510	22.58	71.41
311	17.64	55.77	361	19.00	60.08	411	20.27	64.11	461	21.47	67.90	511	22.61	71.48
312	17.66	55.86	362	19.03	60.17	412	20.30	64.19	462	21.49	67.97	512	22.63	71.55
313	17.69	55.95	363	19.05	60.25	413	20.32	64.27	463	21.52	68.04	513	22.65	71.62
314	17.72	56.04	364	19.08	60.33	414	20.35	64.34	464	21.54	68.12	514	22.67	71.69
315	17.75	56.12	365	19.10	60.42	415	20.37	64.42	465	21.56	68.19	515	22.69	71.76
316	17.78	56.21	366	19.13	60.50	416	20.40	64.50	466	21.59	68.26	516	22.72	71.83
317	17.80	56.30	367	19.16	60.58	417	20.42	64.58	467	21.61	68.34	517	22.74	71.90
318	17.83	56.39	368	19.18	60.66	418	20.45	64.65	468	21.63	68.41	418	22.76	71.97
319	17.86	56.48	369	19.21	60.75	419	20.47	64.73	469	21.66	68.48	519	22.78	72.04
320	17.89	56.57	370	19.24	60.83	420	20.49	64.81	470	21.68	68.56	520	22.80	72.11
321	17.92	56.66	371	19.26	60.91	421	20.52	64.88	471	21.70	68.63	521	22.83	72.18
322	17.94	56.75	372	19.29	60.99	422	20.54	64.96	472	21.73	68.70	522	22.85	72.25
323	17.97	56.83	373	19.31	61.07	423	20.57	65.04	473	21.75	68.78	523	22.87	72.32
324	18.00	56.92	374	19.34	61.16	424	20.59	65.12	474	21.77	68.85	524	22.89	72.39
325	18.03	57.01	375	19.36	61.24	425	20.62	65.19	475	21.79	68.92	525	22.91	72.46
326	18.06	57.10	376	19.39	61.32	426	20.64	65.27	476	21.82	68.99	526	22.93	72.53
327	18.08	57.18	377	19.42	61.40	427	20.66	65.35	477	21.84	69.07	527	22.96	72.59
328	18.11	57.27	378	19.44	61.48	428	20.69	65.42	478	21.86	69.14	528	22.98	72.66
329	18.14	57.36	379	19.47	61.56	429	20.71	65.50	479	21.89	69.21	529	23.00	72.73
330	18.17	57.45	380	19.49	61.64	430	20.74	65.57	480	21.91	69.28	530	23.02	72.80
331	18.19	57.53	381	19.52	61.73	431	20.76	65.65	481	21.93	69.35	531	23.04	72.87
332	18.22	57.62	382	19.54	61.81	432	20.78	65.73	482	21.95	69.43	532	23.07	72.94
333	18.25	57.71	383	19.57	61.89	433	20.81	65.80	483	21.98	69.50	533	23.09	73.01
334	18.28	57.79	384	19.60	61.97	434	20.83	65.88	484	22.00	69.57	534	23.11	73.08
335	18.30	57.88	385	19.62	62.05	435	20.86	65.95	485	22.02	69.64	535	23.13	73.14
336	18.33	57.97	386	19.65	62.13	436	20.88	66.03	486	22.05	69.71	536	23.15	73.21
337	18.36	58.05	387	19.67	62.21	437	20.90	66.11	487	22.07	69.79	537	23.17	73.28
338	18.38	58.14	388	19.70	62.29	438	20.93	66.18	488	22.09	69.86	538	23.19	73.35
339	18.41	58.22	389	19.72	62.37	439	20.95	66.26	489	22.11	69.93	539	23.22	73.42
340	18.44	58.31	390	19.75	62.45	440	20.98	66.33	490	22.14	70.00	540	23.24	73.48
341	18.47	58.40	391	19.77	62.53	441	21.00	66.41	491	22.16	70.07	541	23.26	73.55
342	18.49	58.48	392	19.80	62.61	442	21.02	66.48	492	22.18	70.14	542	23.28	73.62
343	18.52	58.57	393	19.82	62.69	443	21.05	66.56	493	22.20	70.21	543	23.30	73.69
344	18.55	58.65	394	19.85	62.77	444	21.07	66.63	494	22.23	70.29	544	23.32	73.76
345	18.57	58.74	395	19.87	62.85	445	21.10	66.71	495	22.25	70.36	545	23.35	73.82
346	18.60	58.82	396	19.90	62.93	446	21.12	66.78	596	22.27	70.43	546	23.37	73.89
347	18.63	58.91	397	19.92	63.01	447	21.14	66.86	497	22.29	70.50	547	23.39	73.96
348	18.65	58.99	398	19.95	63.09	448	21.17	66.93	498	22.32	70.57	548	23.41	74.03
349	18.68	59.08	399	19.97	63.17	449	21.19	67.01	499	22.34	70.64	549	23.43	74.09

n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
550	23.45	74.16	600	24.49	77.46	650	25.50	80.62	700	26.46	83.67	750	27.39	86.60
551	23.47	74.23	601	24.52	77.52	651	25.51	80.68	701	26.48	83.73	751	27.40	86.66
552	23.49	74.30	602	24.54	77.59	652	25.53	80.75	702	26.50	83.79	752	27.42	86.72
553	23.52	74.36	603	24.56	77.65	653	25.55	80.81	703	26.51	83.85	753	27.44	86.78
554	23.54	74.43	604	24.58	77.72	654	25.57	80.87	704	26.53	83.90	754	27.46	86.83
555	23.56	74.50	605	24.60	77.78	655	25.59	80.93	705	26.55	83.96	755	27.48	86.89
556	23.58	74.57	606	24.62	77.85	656	25.61	80.99	706	26.57	84.02	756	27.50	86.95
557	23.60	74.63	607	24.64	77.91	657	25.63	81.06	707	26.59	84.08	757	27.51	87.01
558	23.62	74.70	608	24.66	77.97	658	25.65	81.12	708	26.61	84.14	758	27.53	87.06
559	23.64	74.77	609	24.68	78.04	659	25.67	81.18	709	26.63	84.20	759	27.55	87.12
560	23.66	74.83	610	24.70	78.10	660	25.69	81.24	710	26.65	84.26	760	27.57	87.18
561	23.69	74.90	611	24.72	78.17	661	25.71	81.30	711	26.66	84.32	761	27.59	87.24
562	23.71	74.97	612	24.74	78.23	662	25.73	81.36	712	26.68	84.38	762	27.60	87.29
563	23.73	75.03	613	24.76	78.29	663	25.75	81.42	713	26.70	84.44	763	27.62	87.35
564	23.75	75.10	614	24.78	78.36	664	25.77	81.49	714	26.72	84.50	764	27.64	87.41
565	23.77	75.17	615	24.80	78.42	665	25.79	81.55	715	26.74	84.56	765	27.66	87.46
566	23.79	75.23	616	24.82	78.49	666	25.81	81.61	716	26.76	84.62	766	27.68	87.52
567	23.81	75.30	617	24.84	78.55	667	25.83	81.67	717	26.78	84.68	767	27.69	87.58
568	23.83	75.37	618	24.86	78.61	668	25.85	81.73	718	26.80	84.73	768	27.71	87.64
569	23.85	75.43	619	24.88	78.68	669	25.87	81.79	719	26.81	84.79	769	27.73	87.69
570	23.87	75.50	620	24.90	78.74	670	25.88	81.85	720	26.83	84.85	770	27.75	87.75
571	23.90	75.56	621	24.92	78.80	671	25.90	81.91	721	26.85	84.91	771	27.77	87.81
572	23.92	75.63	622	24.94	78.87	672	25.92	81.98	722	26.87	84.97	772	27.78	87.86
573	23.94	75.70	623	24.96	78.93	673	25.94	82.04	723	26.89	85.03	773	27.80	87.92
574	23.96	75.76	624	24.98	78.99	674	25.96	82.10	724	26.91	85.09	774	27.82	87.98
575	23.98	75.83	625	25.00	79.06	675	25.98	82.16	725	26.93	85.15	775	27.84	88.03
576	24.00	75.89	626	25.02	79.12	676	26.00	82.22	726	26.94	85.21	776	27.86	88.09
577	24.02	75.96	627	25.04	79.18	677	26.02	82.28	727	26.96	85.26	777	27.87	88.15
578	24.04	76.03	628	25.06	79.25	678	26.04	82.34	728	26.98	85.32	778	27.89	88.20
579	24.06	76.09	629	25.08	79.31	679	26.06	82.40	729	27.00	85.38	779	27.91	88.26
580	24.08	76.16	630	25.10	79.37	680	26.08	82.46	730	27.02	85.44	780	27.93	88.32
581	24.10	76.22	631	25.12	79.44	681	26.10	82.52	731	27.04	85.50	781	27.95	88.37
582	24.12	76.29	632	25.14	79.50	682	26.12	82.58	732	27.06	85.56	782	27.96	88.43
583	24.15	76.35	633	25.16	79.56	683	26.13	82.64	733	27.07	85.62	783	27.98	88.49
584	24.17	76.42	634	25.18	79.62	684	26.15	82.70	734	27.09	85.67	784	28.00	88.54
585	24.19	76.49	635	25.20	79.69	685	26.17	82.76	735	27.11	85.73	785	28.02	88.60
586	24.21	76.55	636	25.22	79.75	686	26.19	82.83	736	27.13	85.79	786	28.04	88.66
587	24.23	76.62	637	25.24	79.81	687	26.21	82.89	737	27.15	85.85	787	28.05	88.71
588	24.25	76.68	638	25.26	79.87	688	26.23	82.95	738	27.17	85.91	788	28.07	88.77
589	24.27	76.75	639	25.28	79.94	689	26.25	83.01	739	27.18	85.97	789	28.09	88.83
590	24.29	76.81	640	25.30	80.00	690	26.27	83.07	740	27.20	86.02	790	28.11	88.88
591	24.31	76.88	641	25.32	80.06	691	26.29	83.13	741	27.22	86.08	791	28.12	88.94
592	24.33	76.94	642	25.34	80.12	692	26.31	83.19	742	27.24	86.14	792	28.14	88.99
593	24.35	77.01	643	25.36	80.19	693	26.32	83.25	743	27.26	86.20	793	28.16	89.05
594	24.37	77.07	644	25.38	80.25	694	26.34	83.31	744	27.28	86.26	794	28.18	89.11
595	24.39	77.14	645	25.40	80.31	695	26.36	83.37	745	27.29	86.31	795	28.20	89.16
596	24.41	77.20	646	25.42	80.37	696	26.38	83.43	746	27.31	86.37	796	28.21	89.22
597	24.43	77.27	647	25.44	80.44	697	26.40	83.49	747	27.33	86.43	797	28.23	89.27
498	24.45	77.33	648	25.46	80.50	698	26.42	83.55	748	27.35	86.49	798	28.25	89.33
599	24.47	77.40	649	25.48	80.56	699	26.44	83.61	749	27.37	86.54	799	28.27	89.39

n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$	n	$\sqrt{n}$	$\sqrt{10n}$
800	28.28	89.44	850	29.15	92.20	900	30.00	94.87	950	30.82	97.47
801	28.30	89.50	851	29.17	92.25	901	30.02	94.92	951	30.84	97.52
802	28.32	89.55	852	29.19	92.30	902	30.03	94.97	952	30.85	97.57
803	28.34	89.61	853	29.21	92.36	903	30.05	95.03	953	30.87	97.62
804	28.35	89.67	854	29.22	92.41	904	30.07	95.08	954	30.89	97.67
805	28.37	89.72	855	29.24	92.47	905	30.08	95.13	955	30.90	97.72
806	28.39	89.78	856	29.26	92.52	906	30.10	95.18	956	30.92	97.78
807	28.41	89.83	857	29.27	92.57	907	30.12	95.24	957	30.94	97.83
808	28.43	89.89	858	29.29	92.63	908	30.13	95.29	958	30.95	97.88
809	28.44	89.94	859	29.31	92.68	909	30.15	95.34	959	30.97	97.93
810	28.46	90.00	860	29.33	92.74	910	30.17	95.39	960	30.98	97.98
811	28.48	90.05	861	29.34	92.79	911	30.18	95.45	961	31.00	98.03
812	28.50	91.11	862	29.36	92.84	912	30.20	95.50	962	31.02	98.08
813	28.51	90.17	863	29.38	92.90	913	30.22	95.55	963	31.03	98.13
814	28.53	90.22	864	29.39	92.95	914	30.23	95.60	964	31.05	98.18
815	28.55	90.28	865	29.41	93.01	915	30.25	95.66	965	31.06	98.23
816	28.57	90.33	866	29.43	93.06	916	30.27	95.71	966	31.08	98.28
817	28.58	90.39	867	29.44	93.11	917	30.28	95.76	967	31.10	98.34
818	28.60	90.44	868	29.46	93.17	918	30.30	95.81	968	31.11	98.39
819	28.62	90.50	869	29.48	93.22	919	30.32	95.86	969	31.13	98.44
820	28.64	90.55	870	29.50	93.27	920	30.33	95.92	970	31.14	98.49
821	28.65	90.61	871	29.51	93.33	921	30.35	95.97	971	31.16	98.54
822	28.67	90.66	872	29.53	93.49	922	30.36	96.02	972	31.18	98.59
823	28.69	90.72	873	29.55	93.43	923	30.38	96.07	973	31.19	98.64
824	28.71	90.77	874	29.56	93.49	924	30.40	96.12	974	31.21	98.69
825	28.72	90.83	875	29.58	93.54	925	30.41	96.18	975	31.22	98.74
826	28.74	90.88	876	29.60	93.59	926	30.43	96.23	976	31.24	98.79
827	28.76	90.94	877	29.61	93.65	927	30.45	96.28	977	31.26	98.84
828	28.77	90.99	878	29.63	93.70	928	30.46	96.33	978	31.27	98.89
829	28.79	91.05	879	29.65	93.76	929	30.48	96.38	979	31.29	98.94
830	28.81	91.10	880	29.66	93.81	930	30.50	96.44	980	31.30	98.99
831	28.83	91.16	881	29.68	93.86	931	30.51	96.49	981	31.32	99.05
832	28.84	91.21	882	29.70	93.91	932	30.53	96.54	982	31.34	99.10
833	28.86	91.27	883	29.72	93.97	933	30.55	96.59	983	31.35	99.15
834	28.88	91.32	884	29.73	94.02	934	30.56	96.64	984	31.37	99.20
835	28.90	91.38	885	29.75	94.07	935	30.58	96.70	985	31.38	99.25
836	28.91	91.43	886	29.77	94.13	936	30.59	96.75	986	31.40	99.30
837	28.93	91.49	887	29.78	94.18	937	30.61	96.80	987	31.42	99.35
838	28.95	91.54	888	29.80	94.23	938	30.63	96.85	988	31.43	99.40
839	28.97	91.60	889	29.82	94.29	939	30.64	96.90	989	31.45	99.45
840	28.98	91.65	890	29.83	94.34	940	30.66	96.95	990	31.46	99.50
841	29.00	91.71	891	29.85	94.39	941	30.68	97.01	991	31.48	99.55
842	29.02	91.76	892	29.87	94.45	942	30.69	97.06	992	31.50	99.60
843	29.03	91.82	893	29.88	94.50	943	30.71	97.11	993	31.51	99.65
844	29.05	91.87	894	29.90	94.55	944	30.72	97.16	994	31.53	99.70
845	29.07	91.92	895	29.92	94.60	945	30.74	97.21	995	31.54	99.75
846	29.09	91.98	896	29.93	94.66	946	30.76	97.26	996	31.56	99.80
847	29.10	92.03	897	29.95	94.71	947	30.77	97.31	997	31.58	99.85
848	29.12	92.09	898	29.97	94.76	948	30.79	97.37	998	31.59	99.90
849	29.14	92.14	899	29.98	94.82	949	30.81	97.42	999	31.61	99.95
									1000	31.62	100.0

TABLE OF TRIGONOMETRIC FUNCTIONS

Degrees	Radians	Cosine	Sine	Tangent	Degrees	Radians	Cosine	Sine	Tangent
0	.000	1.000	0.000	0.000					
1	.017	1.000	.018	.018	46	.803	.695	.719	1.036
2	.035	0.999	.035	.035	47	.820	.682	.731	1.072
3	.052	.999	.052	.052	48	.838	.669	.743	1.111
4	.070	.998	.070	.070	49	.855	.656	.755	1.150
5	.087	.996	.087	.087	50	.873	.643	.766	1.192
6	.105	.995	.105	.105	51	.890	.629	.777	1.235
7	.122	.993	.122	.123	52	.908	.616	.788	1.280
8	.140	.990	.139	.141	53	.925	.602	.799	1.327
9	.157	.988	.156	.158	54	.942	.588	.809	1.376
10	.175	.985	.174	.176	55	.960	.574	.819	1.428
11	.192	.982	.191	.194	56	.977	.559	.829	1.483
12	.209	.978	.208	.213	57	.995	.545	.839	1.540
13	.227	.974	.225	.231	58	1.012	.530	.848	1.600
14	.244	.970	.242	.249	59	1.030	.515	.857	1.664
15	.262	.966	.259	.268	60	1.047	.500	.866	1.732
16	.279	.961	.276	.287	61	1.065	.485	.875	1.804
17	.297	.956	.292	.306	62	1.082	.470	.883	1.881
18	.314	.951	.309	.325	63	1.100	.454	.891	1.963
19	.332	.946	.326	.344	64	1.117	.438	.899	2.050
20	.349	.940	.342	.364	65	1.134	.423	.906	2.145
21	.367	.934	.358	.384	66	1.152	.407	.914	2.246
22	.384	.927	.375	.404	67	1.169	.391	.921	2.356
23	.401	.921	.391	.424	68	1.187	.375	.927	2.475
24	.419	.914	.407	.445	69	1.204	.358	.934	2.605
25	.436	.906	.423	.466	70	1.222	.342	.940	2.747
26	.454	.899	.438	.488	71	1.239	.326	.946	2.904
27	.471	.891	.454	.510	72	1.257	.309	.951	3.078
28	.489	.883	.470	.532	73	1.274	.292	.956	3.271
29	.506	.875	.485	.554	74	1.292	.276	.961	3.487
30	.524	.866	.500	.577	75	1.309	.259	.966	3.732
31	.541	.857	.515	.601	76	1.326	.242	.970	4.011
32	.559	.848	.530	.625	77	1.344	.225	.974	4.331
33	.576	.839	.545	.649	78	1.361	.208	.978	4.705
34	.593	.829	.559	.675	79	1.379	.191	.982	5.145
35	.611	.819	.574	.700	80	1.396	.174	.985	5.671
36	.628	.809	.588	.727	81	1.414	.156	.988	6.314
37	.646	.799	.602	.754	82	1.431	.139	.990	7.115
38	.663	.788	.616	.781	83	1.449	.122	.993	8.144
39	.681	.777	.629	.810	84	1.466	.105	.995	9.514
40	.698	.766	.643	.839	85	1.484	.087	.996	11.43
41	.716	.755	.656	.869	86	1.501	.070	.998	14.30
42	.733	.743	.669	.900	87	1.518	.052	.999	19.08
43	.750	.731	.682	.933	88	1.536	.035	.999	28.64
44	.768	.719	.695	.966	89	1.553	.018	1.000	57.29
45	.785	.707	.707	1.000	90	1.571	.000	1.000	unde-fined

ERIC